TORSIONAL VIBRATIONS AND STABILITY OF THIN-WALLED BEAMS OF OPEN SECTION RESTING ON CONTINUOUS ELASTIC FOUNDATION

A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY IN MECHANICAL ENGINEERING

By
CHELLAPILLA KAMESWARA RAO, B.SC., B.E., M.E.

DEPARTMENT OF MECHANICAL ENGINEERING
COLLEGE OF ENGINEERING
ANDHRA UNIVERSITY
WALTAIR 530 013
AUGUST 1975

DEDICATED
IN LOVING MEMORY
TO
Late Professor K. VENKATA APPA RAO
Head of the Department of Mechanical Engineering
Principal, College of Engineering
ANDHRA UNIVERSITY, WALTAIR.

DECLARATION

I declare that this work is entirely original and has not been submitted in part or full for any Degree, Diploma or Title of any other University.

CKameswara Rao

WALTAIR,
AUGUST, 1975.

(C. KAMESWARA RAO)
AUTHOR.

CERTIFICATE

This is to certify that this thesis is the original work done by Sri Challapilla Kameswara Rao, B.Sc., B.E., M.E., Associate Lecturer, Department of Mechanical Engineering, Andhra University, Waltair and that this thesis has not been submitted in part or full for any degree or diploma of any other University.

(Prof.P.K.SARMA)
Guide And Supervisor.

BIOGRAPHICAL SKETCH.

The author was born in Peddapuram Bhimavaram, East West Godavary District, Andhra Pradesh, on 27th December 1947, and was the eldest son of Chellapilla Bangareswara Sarma, Head Master, Municipal High School, Narasaraopet, Guntur District, Andhra Pradesh. He was educated in Narasaraopet Municipal High School and S.S.N.College and obtained Bachelor of Science Degree from Andhra University in 1965. In 1968, he received the Bachelor's Degree in Mechanical Engineering from Sri Venkateswara University. He received the Master's Degree in Machine Design from Andhra University in February 1971.

He worked as Junior Research Fellow (C.S.I.R.) and Senior Research Fellow (U.G.C.) during 1970-'71 and joined the Faculty of Mechanical Engineering, Andhra University College of Engineering, Waltair in December 1971, as Associate Lecturer. He registered for Ph.D. in 1972 and worked under the guidance of Late Professor K.V.Appa Rao. Upon his sudden demise, he completed his Doctoral thesis work under the guidance of Prof.P.K.Sarma.

The author was married in May 1970, and has two sons, Hema Chandra Kumar and Suresh Babu.

TABLE OF CONTENTS

Page No.

Acknowledgements. .. i

Nomenclature. .. ii

Abstract. .. viii

List of Figures. .. xiv

List of Tables. .. xix

Chapter - I: INTRODUCTION AND REVIEW OF RELEVENT LITERATURE.

1.1 Introduction. .. 1

1.2 Brief review of relevent literature. ..

1.2.1 Elastic Stability. .. 3

1.2.2 Vibrations and Wave propagation. .. 8

1.3 Aim and scope of the present investigation. .. 19

Chapter - II: TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS ON ELASTIC FOUNDATION - AN EXACT ANALYSIS.

2.1 Introduction. ..23

2.2 Basic Assumptions. .25

2.3 Derivation of basic differential equation. ..26

2.4 (a) Natural boundary conditions. .31

(b) Time-dependent boundary conditions. ..33

2.5 Analysis of various terms. .34

2.6 Non-dimensionalization and general solution of equation of motion. .35

2.7 Frequency equations and modal functions. .38

2.7.1 Simply supported beam. .49
2.7.2 Fixed-end beam. .51
2.7.3 Beam fixed at one end, simply supported at the other. .52
2.7.4 Cantilever beam with warping restrained. .52
2.7.5 Cantilever beam with unrestrained warping. 53
2.7.6 Beam with free ends. .54
2.8 Results and discussion. .55
2.9 Approximate solutions by Galerkin's technique. .58
2.9.1 Fixed-end beam. .63
2.9.2 Beam fixed at one end and simply supported at the other. .64
2.9.3 Limiting conditions. .65
2.10 Remarks. .68

Chapter - III: FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS ON ELASTIC FOUNDATION.

3.1 Introduction. .69
3.2 Finite Element Concept. .71
3.3 Basic Finite Element theory for vibrations. .73
3.4 (a) Functional representation of angle of twist. 77
(b) Formulation of Element matrices. 81
3.5 Equations of equilibrium for the totally assembled beam. .85
3.6 Boundary Conditions. .87
3.7 Method Solution. .88
3.8 Results and Conclusions. .89

Chapter - IV: EFFECT OF LONGITUDINAL INERTIA AND OF SHEAR DEFORMATION ON THE TORSIONAL FREQUENCIES AND NORMAL MODES OF SHORT WIDE-FLANGED THIN-WALLED BEAMS OF OPEN SECTION.

4.1 Introduction. .103

4.2 Basic Assumptions. .106

4.3 Derivation of differential equations of motion .106

4.4 (a) Natural boundary conditions. .112

(b) Time-dependent boundary conditions. .114

4.5.1 Single equation in angle of twist. .114

4.5.2 Analysis of various terms. .115

4.6 Non-dimensionalization and General solution. .117

4.7 Frequency equations and Modal functions. .120

- 4.7.1 Simply supported beam. .121
- 4.7.2 Fixed-fixed beam. .126
- 4.7.3 Beam fixed at one end and simply supported at the other. .128
- 4.7.4 Cantilever beam with one end fixed and free at the other. .129
- 4.7.5 Cantilever beam with one end simply supported and free at the other. ..131
- 4.7.6 Beam with free ends. ..132

4.8 Orthoganality and Normalizing conditions. ..133

4.9 Approximate solutions by Galerkin's technique. ..140

- 4.9.1 Fixed-fixed beam. ..140
- 4.9.2 Beam fixed at one end and simply supported at the other. ..141

4.10 Results and Conclusions. ..143

Chapter - V: FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS OF SHORT WIDE-FLANGED THIN-WALLED BEAMS INCLUDING THE EFFECTS OF LONGITUDINAL INERTIA AND SHEAR DEFORMATION.

5.1 Introduction. ..156

5.2 Modified energy expressions! .157

5.3 Modified natural boundary conditions. ..160

5.4 Finite element formulation. ..161

5.5 Derivation of element matrices including Second order effects. ..164

5.6 Equations of equilibrium for the totally assembled beam. ..169

5.7 Boundary conditions useful for modifying the total matrices. ..170

5.8 Results and conclusions. ..171

Chapter - VI: FORCED TORSIONAL VIBRATIONS OF SHORT WIDE-FLANGED BEAMS WITH LONGITUDINAL INERTIA, SHEAR DEFORMATION AND VISCOUS DAMPING.

6.1 Introduction. ·175

6.2 Derivation of equations of motion including viscous damping. ·176

6.3 Solution for the case of a simply supported beam. ·179

6.4 Response to a uniformly distributed torsional forcing function sinusoidal in time. ·183

6.5 Free and forced vibrations of a classic beam simply supported at both ends. 185

6.6 Discussion of numerical results. .187

Chapter - VII: TORSIONAL WAVE PROPAGATION IN ORTHOTROPIC THIN-WALLED BEAMS OF OPEN SECTION INCLUDING THE EFFECTS OF LONGITUDINAL INERTIA AND SHEAR DEFORMATION.

7.1 Introduction. ..198

7.2 Analysis and Examples. ..200

7.3 Results and discussion. ..207

Chapter - VIII: TORSIONAL VIBRATIONS AND STABILITY OF SHORT THIN-WALLED BEAMS RESTING ON CONTINUOUS ELASTIC FOUNDATION.

8.1 Introduction. ..218

8.2 Derivation of coupled equations of motion including axial load and elastic foundation. ..219

8.3 Natural boundary conditions. ..221

8.4.1 Single equation in angle of twist. ..222

8.4.2 Analysis of various terms. ..223

8.5 Non-dimensionalization and general solution. ..224

8.6 Frequency or buckling load equations and modal functions. ..228

- 8.6.1 Simply supported beam. ..228
- 8.6.2 Fixed-fixed beam ..232
- 8.6.3 Beam fixed at one end and simply supported at the other. ..234
- 8.6.4 Cantilever beam with one end fixed and free at the other. ..235
- 8.6.5 Cantilever beam with one end simply supported and free at the other. ..237
- 8.6.6 Beam with free ends. ..238

8.7 Approximate solutions by Galerkin's technique. ..239

- 8.7.1 Fixed-fixed beam. ..239
- 8.7.2 Beam fixed at one end and simply supported at the other. ..241

8.8 Limiting conditions. .. 24a

8.9 Results and conclusions. .. 241

Chapter - IX: FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS AND STABILITY OF SHORT THIN-WALLED BEAMS RESTING ON CONTINUOUS ELASTIC FOUNDATION.

9.1 Introduction. .. 263

9.2 Modified strain energy expression including the effects of axial load and elastic foundation. .. 264

9.3 Modified natural boundary conditions .. 265

9.4 Derivation of element matrices including axial load, elastic foundation and second order effects. .. 265

9.5 Results and conclusions. .. 270

Chapter - X: NON-LINEAR TORSIONAL STABILITY OF LENGTHY THIN-WALLED BEAMS OF OPEN SECTION RESTING ON CONTINUOUS ELASTIC FOUNDATION.

10.1 Introduction. .. 274

10.2 Development of governing differential equation and boundary conditions. .. 275

10.3 Simply supported beam. .. 279

10.4 Clamped beam. .. 281

10.5 Results and conclusions. .. 283

Chapter - XI: NON-LINEAR TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS OF OPEN SECTION RESTING ON CONTINUOUS ELASTIC FOUNDATION.

11.1 Introduction. .. 287

11.2 Derivation of governing differential equation .. 289

11.3 Simply supported beam. .. 289
11.4 Clamped beam. .. 292
11.5 Results and conclusions. .. 294

Chapter - XII: SUMMARY AND RECOMMENDATIONS.

12.1 Conclusions. ..307
12.2 Recommendations for future work. ..314
REFERENCES. .. 315

ACKNOWLEDGEMENTS

The author wishes to express his deep appreciation to Dr.P.K.Sarma, Ph.D.(Moscow), Professor of Mechanical Engineering for his supervision and encouragement throughout the course of the research work. The author sincerely thanks Prof.T.Venugopala Rao, B.E., M.S.(Illinois), A.M.I.E., Head of the Department of Mechanical Engineering for the facilities provided. The author wishes to acknowledge the services of Sri Ramaiah in typing the manuscript and Sri E.Satyanarayana in preparing the graphs. The author thanks the Board of Research Studies, Andhra University, Waltair for having given free computer time.

Last but not the least, the author specially thanks his wife Smt. Venkata Ramana for her patience, understanding and forbearance throughout the period of this investigation.

NOMENCLATURE.

Dimensions and Sectional Properties:

A = total area of cross section

A_f = area of each flange

b_f = width of ~~the bar~~ each flange

C_s = torsion constant

C_w = warping constant

F = constant depending upon cross sectional properties, see Eq.(10.4)

h = height between the centerlines of the flanges

I_f = moment of inertia of each flange about y-axis

I_p = polar moment of inertia of the cross section

I_R = fourth moment of inertia about the shear center, see Eq.(10.5)

I_{pc} = half the polar moment of inertia about the shear center

K' = numerical shape factor for cross section

L = length of the beam

t_f = thickness of each flange

t_w = thickness of the web

S_o = statical moment with respect to neutral axis

z = displacement along the length of the bar

Material Properties:

E = Young's modulus

E_{zz} = modulus for extension-compression along the axis of the bar

G = shear modulus

G_{zx} = shear modulus of orthotropic material
K_t = foundation modulus in torsion
ρ = mass density of the material of the beam

Forces, displacements and Moments:

M = moment in each flange
M_y = net bending moment in the cross section
P = axial compressive load
P_{cr} = torsional buckling load
P^* = post-buckling load
q = external viscous force per unit length acting along the sides of the flanges opposing warping
Q = shear force due to bending in the flanges
T_e = external torque per unit length of the beam
T_o = a constant equal to the static torque
T_s = torsional couple
T_t = $T_s + T_w$ = total torque
T_w = warping torque
u = x-displacement of the top flange center line
w = z-displacement of a point in the top flange
ϕ = angle of twist
ϕ = normal function of ϕ
ϕ_s = contribution of shear strain to the angle of twist
ϕ_t = angle of twist when shear strain has been neglected
ψ = warping angle
$\bar{\psi}$ = normal function of ψ

Stresses and Strains:

σ_x, σ_y, σ_z = normal stresses in x, y and z directions respectively

τ_{zx} = maximum shear stress in flange bending

ϵ_{sh} = shear strain at the center of the flange, x=0

ϵ_z = z-component of strain

Energies and Matrices:

$\bar{A}$ = transformation matrix for displacements whose elements are functions of x, y and z

$\bar{C}$ = transformation matrix giving the strains in terms of generalized displacements

$\bar{D}$ = matrix of material constants

$\bar{F}$ = total load matrix

$\bar{K}$ = total stiffness matrix

$\bar{M}$ = total mass matrix

$\bar{q}$, $\bar{R}$ = column matrices of generalized displacements

$\bar{Q}$, $\bar{r}$ = column vectors of amplitudes of generalized displacements

$\bar{S}$ = total stability coefficient matrix

T_k = kinetic energy of the strained bar

u = components of the displacement vector

U = total strain energy

W = potential energy

$\bar{\sigma}$ = matrix of stresses

$\bar{\epsilon}$ = matrix of strains

Non-dimensional Parameters:

$\bar{a}^2$ $= 1+s^2K^2 - K^2/\lambda^2 d^2$

d^2 $= I_f h^2/2\,I_p L^2$ = longitudinal inertia parameter

K^2 $= GC_s L^2/EC_w$ = warping parameter

r_n = ratios of eigen values (n=1 to 4)

s^2 $= EI_f/K'A_f GL^2$ = shear parameter

$\bar{t}_1$ $= (EC_w/\rho I_p L^4)^{1/2}\,t$ = dimensionless time

z $= z/L$ = non-dimensional beam length

$\bar{\alpha}_3$ $= E_{zz}/G_{zx}$

$\bar{\beta}_3$ $= (C_s + 1/2\,K'A_f h^2)/\,I_p$

$\bar{\eta}\,\bar{g}$ $= K'A_f h^2/I_f$

$\bar{\varepsilon}_2$ $= C_s/I_p$

Δ^2 $= PI_p L^2/AEC_w$ = axial load parameter

Δ^2_{cr} $= P_{cr} I_p L^2/AEC_w$ = torsional buckling load parameter

Δ^{*2}_{cr} $= P^* I_p L^2/AEC_w$ = post-buckling load parameter

γ^2 $= K_t L^4/4EC_w$ = foundation parameter

λ^2_c $= 1/s^2 d^2$ = critical frequency parameter

λ^2_n $= \rho I_p L^4 p_n^2/EC_w$ = frequency parameter

Ω^2 $= 2I_f/I_p$

δ^* = P/C_w

Miscellaneous:

c_o = bar velocity = $(E_{zz}/\rho)^{1/2}$

c_2 = shear wave velocity = $(G_{zx}/\rho)^{1/2}$

c_p = phase velocity for torsional waves

i = $\sqrt{-1}$

n = mode number

N = Number of segments into which the beam is subdevided

p_n = natural frequency of vibration in radious per unit time.

t = time

T = linear period of torsional vibration

T^* = non-linear period of torsional vibration

X = normal function giving the shape of mode of vibration

α_n, α'_n, β_n = positive real quantities ($n=1,2,3$)

β^* = torsional amplitude in non-linear analysis

β_t = torsional damping constant

β_w = warping damping constant

τ_n = torsional excitation function

τ^* = a function of time in non-linear analysis

ϵ^* = error function

δ = variational operator

δ_1 = wave number = $2\pi/\lambda$

ω = torsional excitation frequency

Λ = wavelength

Salient symbols are listed above. Other symbols are defined in the body of the thesis as and when they appear.

ABSTRACT

This thesis presents some analytical studies of linear and non-linear torsional vibrations and stability of uniform thin-walled beams of open section resting on continuous elastic foundation subjected to a time-invariant axial compressive load including the effects of longitudinal inertia and shear deformation.

Based on the Timoshenko torsion theory, the problem of linear torsional vibrations and stability of uniform lengthy thin-walled beams of open section resting on continuous elastic foundation subjected to a time-invariant axial compressive load is analyzed exactly by using the method of separation of variables. The frequency or buckling load and normal mode equations are derived for various end conditions. Approximate expressions are derived for the torsional frequency and buckling loads using Galerkin's technique. The results presented for some typical boundary conditions reveal that for lower modes, the increase in the foundation parameter increases the frequency parameter significantly and the increase in the axial load parameter decreases the frequency parameter considerably. The combined influence of axial load and foundation parameters is observed to be the superimposition of the individual effects on the frequency of vibration.

Finite element formulation of the problem of free torsional vibrations of thin-walled beams of open section resting on continuous elastic foundation is also presented. The stiffness and consistent mass matrices are derived and the eigen value problem

is formulated. The eigen values obtained by finite-element method compared favourably well with the exact values even for a coarse subdivision of the beam into six elements. A digital computer programme is written for obtaining the results for the frequency parameter for various boundary conditions.

As the corrections due to second order effects may be of importance if the effect of cross sectional dimensions on frequencies of vibration are desired, an exact analysis is presented for free torsional vibrations of short thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. New frequency and normal mode equations are derived for six common types of simple and finite beams. Solutions of the frequency equations for some typical boundary conditions are obtained on a digital computer. The individual effects of longitudinal inertia and shear deformation on the torsional frequencies of a simply supported beam are shown graphically. The torsional frequency values and the modifying quotients for the first four modes of vibration for some typical boundary conditions are presented in tabular form suitable for design use; They showing the combined effects of longitudinal inertia and shear deformation. Approximate frequency equations for some typical end conditions are obtained using Galerkin's technique. It is observed that the effect of shear deformation is to decrease the stiffness of the beam and thus results in corresponding decrease of natural frequencies. The decrease is relatively small compared to the increase due to warping; however, the importance of shear deformation appears when higher frequencies are considered.

A finite-element formulation of the problem of free-torsional vibrations of short thin-walled beams of open section including the effects of longitudinal inertia and shear deformation is also presented. The corresponding stiffness and mass matrices including these second order effects are derived. The eigen values obtained by the finite element method compared very well with the exact values even for a coarse sub-division of the beam into three elements. A digital computer programme is written for obtaining the results for the frequencies and mode shapes for various end conditions.

The problem of forced torsional vibrations of thin-walled beams of open section is studied including the effects of longitudinal inertia and shear deformation. Viscous damping forces arising separately from torsional and warping velocities are included. The two coupled, fundamental equations of motion are formulated in terms of angle of twist and warping angle. The method of solution is demonstrated for arbitrary external torque for the beam having both ends simply-supported. Numerical results are presented for the case when the torque is uniform over the span and varies sinusoidally in time. Amplitude response is plotted against torsional excitation frequency for varying amounts of torsional and warping damping and is compared to the response for the classic beam for the first five symmetric mode shapes. The amplitudes for the thin-walled beam including shear deformation and longitudinal inertia are found to be considerably larger.

As the increased utilization of composite materials in structural applications has made their analysis ever more important, the problem of torsional wave propagation in orthotropic thin-walled beams of open section including longitudinal inertia and shear deformation is solved. The equation for free torsional vibrations of thin-walled beams of open section of orthotropic material including the effects of longitudinal inertia and shear deformation is established analogous to that for isotropic materials. Many fiber-reinforced plastics and pyrolytic-graphite type materials which are mostly in use, are orthotropic or transversely isotropic in the sense that the ratio of in-plane modulus of elasticity to shear modulus is large. It is shown that, for these materials, the corrections due to longitudinal inertia and shear deformation may be of one order of magnitude greater than the corrections in the isotropic case. Graphs are given of the phase velocity versus inverse wavelength for various aspect ratios of beams of different materials.

The problem of torsional vibrations and stability of short thin-walled beams of open section resting on continuous elastic foundation and subjected to an axial compressive load including the effects of longitudinal inertia and shear deformation is solved by means of an exact analysis. Results for buckling loads for various boundary conditions are presented in tabular form showing the effects of shear deformation. The values of torsional frequency parameter for the first four modes of vibration for various boundary conditions and non-dimensional parameters are presented in tabular form suitable for design use. This problem is also solved by means of finite-element method and an excellent agreement is observed between the results from exact analysis and those from the finite-element method.

It is very well known that a large number of problems of torsional vibrations and stability of thin-walled beams arising in modern high speed aircraft structures, missiles and launching vehicles cannot be adequately explained by the classical linear theories alone, since the torsional deformations of these beams are usually of such a magnitude that the assumption of small rotations of cross sections will no longer be valid.

In view of this, an attempt has been made further in this thesis to derive and solve the governing differential equation of large amplitude torsional stability of lengthy thin-walled beams of open section resting on continuous elastic foundation. Graphs indicating the combined influence of large amplitude and foundation parameter on the torsional post-buckling loads for simply supported and clamped beams are presented. Including the effects of axial compressive load and elastic foundation, the problem of non-linear torsional vibration and post-buckling behavior of thin-walled beams resting on continuous elastic foundation is also investigated.

LIST OF FIGURES

Sl. No.	Fig.No.	Description	Page No.
1.	Fig. 2.1	Differential element of an I-beam.	..28
2.	Fig. 2.2	Variation of r_1 with Δ, for values of K = 0.1, 1.0 and 10.0 (n = 1).	..44
3.	Fig. 2.3	Variation of r_2 with γ, for K = 1.0 for the first three modes of vibration.	45
4.	Fig. 2.4	Variation of r_2 with γ, for K = 10.0 for the first three modes of vibration.	..46
5.	Fig. 2.5	Variation of r_3 with Δ and γ, for K=1.0 for the first fundamental mode of vibration.	..48
6.	Fig. 2.6	Variation of r_3 with Δ and γ, for K=10.0 for the first fundamental mode of vibration.	..49
7.	Fig. 2.7	Variation of r_4 with γ for values of K=0.1, 1.0 and 10.0.	..50
8.	Fig. 2.8	Values of frequency & critical Buckling load parameters for a simply supported beam (K=1, n=1).	..56
9.	Fig. 2.9	Values of frequency & critical buckling load parameters for a simply supported beam (K=10, n=1).	..57
10.	Fig. 2.10	Values of frequency & critical Buckling load parameters for a fixed-fixed beam (K=1, n=1).	..59

11. Fig. 2.11 Values of frequency & critical Buckling load parameters for a fixed-fixed beam (K=10, n=1). ..60

12. Fig. 2.12 Values of frequency & critical Buckling load parameters for a fixed-simply supported beam (K=1, n=1). .. 61

13. Fig. 2.13 Values of frequency & critical buckling loads for a fixed-simply supported beam (K=10, n=1). .. 62

14. Fig. 4.1 Geometry and forces on a differential element I section. ..107

15. Fig. 4.2 Strained State of the element (a) Front View (b) Top view. ..108

16. Fig. 4.3 Corrections in natural frequencies of a simply supported beam owing to longitudinal inertia for the first three modes of vibration (s=0). ..145

17. Fig. 4.4 Corrections in natural frequencies of a simply supported beam owing to shear deformation (d=0). ..146

18. Fig. 6.1 Strained state of a Beam Element. ..177

19. Fig. 6.2 Classical beam - Timoshenko Torsion Theory (n=1). ..189

20. Fig. 6.3 Present Analysis ($\beta_1 = 10^3$, n=1). .190

21. Fig. 6.4 Present Analysis ($\beta_1 = 10^4$, n=1). .191

22. Fig. 6.5 Present Analysis ($\beta_1 = 5 \times 10^4$, n=1) ..192

23. Fig. 6.6 Present Analysis ($\beta_o = 0$, n=1). ..193

24. Fig. 6.7 Present Analysis ($\beta_o = 400$, n=1) ..194

25. Fig. 6.8 Present Analysis ($\beta_o = 800$, n=1) ..195

26. Fig. 7.1 Phase velocities for Torsional Waves in I-Beams ($b/h = 0.50$; $t_f/h = 0.040$ $t_w/h = 0.025$). ..208

27. Fig. 7.2 Phase velocities for Torsional Waves in I-Beams ($b/h = 0.50$; $t_f/h = 0.040$ $t_w/h = 0.025$). ..209

28. Fig. 7.3 Phase Velocities for Torsional Waves in I-Beams ($b/h = 0.25$; $t_f/h = 0.025$ $t_w/h = 0.020$). ..210

29. Fig. 7.4 Phase Velocities for Torsional Waves in I-Beams ($b/h = 0.25$; $t_f/h = 0.025$; $t_w/h = 0.020$). ..211

30. Fig. 7.5 Phase Velocities for Torsional Waves in I-Beams ($b/h = 0.75$; $t_f/h = 0.050$; $t_w/h = 0.040$). ..212

31. Fig. 7.6 Phase Velocities for Torsional Waves in I-Beams ($b/h = 0.75$; $t_f/h = 0.050$; $t_w/h = 0.040$). ..214

32. Fig. 7.7 Phase Velocities for Torsional Waves in I-Beams ($b/h = 1.00$; $t_f/h=0.10$; $t_w/h = 0.050$). ..215

33. Fig. 7.8 Phase Velocities for Torsional Waves in I-Beams ($b/h = 1.00$; $t_f/h = 0.10$; $t_w/h = 0.050$). ..216

34. Fig. 10.1 Effect of elastic foundation on the torsional post-buckling behavior of simply supported thin-walled beams of open section. .. 284

35. Fig. 10.2 Effect of elastic foundation on the torsional post-buckling behavior of clamped thin-walled beams of open section. .. 285

36. Fig. 11.1 Influence of Large Amplitudes on period of Non-linear torsional vibration of a steel I-beam. .. 295

37. Fig. 11.2 Influence of Large Amplitudes on period of Non-linear torsional vibration of 36 WF 230 standard steel I-beam. .. 296

38. Fig. 11.3 Influence of Large Amplitudes on period of Non-linear torsional vibration of 24 WF 130 standard steel I-beam. .. 297

39. Fig. 11.4 Influence of Large Amplitudes on period of Non-linear torsional vibration of clamped I-beams. .. 298

40. Fig. 11.5 Influence of Large amplitudes on period of Non-linear torsional vibration of a simply supported I-beam for various values of $\triangle$ and $\bar{\gamma} = 0.0$ .. 299

41. Fig. 11.6 Influence of Large Amplitudes on period of Non-linear torsional vibration of a simply supported I-beam for values of γ and $\Delta = 2.0$. .. 301

42. Fig. 11.7 Influence of Large Amplitudes on period of Non-linear torsional vibration of a Simply Supported I-beam for various values of γ and $\Delta = 6.0$ .. 302

43. Fig. 11.8 Influence of Large Amplitudes on period of Non-linear torsional vibration of a clamped I-beam for various values of Δ and $\gamma = 0.0$. ..303

44. Fig. 11.9 Influence of Large Amplitudes on period of Non-linear torsional vibration of a clamped I-beam for various values of γ and $\Delta = 2.0$. .. 304

45. Fig. 11.10 Influence of Large Amplitudes on period of Non-linear torsional vibration of a clamped beam for various values of γ and Δ 6.0. .. 305

LIST OF TABLES

Sl. No.	Table No.	Description.	Page No.
1.	Table 2.1	Comparison between exact and approximate values of λ^2 for the first mode of vibration of fixed-fixed thin-walled beams (K = 1.0).	.. 66
2.	Table 2.2	Comparison between exact and approximate values of λ^2 for the first mode of vibration of fixed-simply supported thin-walled beams (K = 1.0).	.. 67
3.	Table 3.1	Values of the frequency parameter λ^2 for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter γ for a value of warping parameter K = 1.	.. 90
4.	Table 3.2	Values of the frequency parameter λ^2 for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter γ for a value of warping parameter K = 10.	.. 92
5.	Table 3.3	Values of the frequency parameter λ^2 for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter γ for a value of warping parameter K=0.01.	94

6. Table 3.4 Values of the frequency parameter λ^2 for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter γ for a value of warping parameter K = 0.100. .. 95

7. Table 3.5 Values of the frequency parameter λ^2 for fixed-fixed thin-walled beams of open section on Elastic foundation for various values of foundation and warping parameters γ and K respectively (N = 6). .. 96

8. Table 3.6 Values of the frequency parameter λ^2 for free-free thin-walled beams of open section on Elastic foundation for various values of foundation and warping parameters γ and K respectively (N = 6). .. 97

9. Table 3.7 Values of the frequency parameter λ^2 for simply supported beams for various values of axial load parameter Δ and foundation parameter γ for a value of K = 1. .. 99

10. Table 3.8 Values of the frequency parameter λ^2 for fixed-fixed beams for various values of axial load parameter Δ and foundation parameter γ for a value of K = 1. ..100

11. Table 3.9 Values of the frequency parameter λ^2 for fixed-simply supported beams for various values of axial load parameter $\triangle$ and foundation parameter γ for a value of K = 1. ..101

12. Table 4.1 Effects of Longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams (s = 2d). ..149

13. Table 4.2 Effects of Longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported thin-walled beams (s = 2d). ..150

14. Table 4.3 Effects of Longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped thin-walled beams (s = 2d). ..151

15. Table 4.4 Values of the second set of first four torsional frequencies of simply supported thin-walled beams (s = 2d). ..152

16. Table 4.5 Values of the Second set of first four torsional frequencies of clamped-simply supported thin-walled beams (s = 2d). ..153

17. Table 4.6 Values of the Second set of first four torsional frequencies of clamped-clamped thin-walled beams (s = 2d). ..154

18. Table 5.1 Comparison of first and second sets of values of λ^2 from the Finite element method and those from exact analysis given in Chapter IV for a simply supported beam ($K = 1.541$, $s = 0.046$, $d = 0.023$). ..172

19. Table 5.2 Comparison of the first set of values of λ^2 from the Finite element method and those from exact analysis given in Chapter IV for a fixed-fixed beam ($K = 1.541$, $s = 0.046$, $d = 0.023$). ..173

20. Table 6.1 Values of the natural frequencies and maximum total torsional amplitudes for various modes of vibration of a simply supported beam. ..196

21. Table 8.1 Effects of shear deformation and elastic foundation on the torsional buckling loads of simply supported, clamped-clamped and clamped-simply supported thin-walled beams of open section. ..245

22. Table 8.2 Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams ($\gamma = 0$, $K = 0.01$ and $s = 2d$) ..246

23. Table 8.3 Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams ($\Delta = 0$, $K = 0.01$, $s = 2d$). ..247

24. Table 8.4 Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of simply supported short thin-walled beams (K=0.01, s = 2d). ..24

25. Table 8.5 Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of simply supported short thin-walled beams (K = 0.01, s = 2d). .. 24

26. Table 8.6 Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported thin-walled beams (γ = 0, K = 0.01, s=2d). 250

27. Table 8.7 Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported short thin-walled beams (Δ = 0, K=0.01, s = 2d). ..25

28. Table 8.8 Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of clamped-simply supported short thin-walled beams ($K = 0.01$, $s = 2d$). . 252

29. Table 8.9 Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of clamped-simply supported short thin-walled beams ($K = 0.01$, $s = 2d$). . 253

30. Table 8.10 Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled beams ($\gamma = 0$, $K = 0.01$, $s=2d$). . 254

31. Table 8.11 Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled beams ($\Delta = 0$, $K=0.01$, $s=2d$) .. 255

32. Table 8.12 Combined effects of axial compressive load and elastic foundation in combination with the longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first

set) of clamped-clamped short thin-walled beams (K = 0.01, s = 2d). ..256

33. Table 8.13 Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of clamped-clamped short thin-walled beams (K = 0.01, s = 2d). ..257

34. Table 8.14 Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams (s = 0.10, d = 0.05). ..258

35. Table 8.15 Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported short thin-walled beams (s = 0.10, d = 0.05). ..259

36. Table 8.16 Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin walled beams(s = 0.10 and d=0.05).260

37. Table 9.1 Comparison of first set of values of λ^2 for various values of $\triangle$ and γ from the Finite Element method and those from exact analysis given in Chapter VIII for a simply supported beam (K = 1.541, s = 0.046, d = 0.023). ..271

38. Table 9.2 Comparison of second set of values of λ^2 for various values of $\triangle$ and γ from the Finite Element method and those from exact analysis given in Chapter VIII for a simply supported beam (K = 1.541, s = 0.046, d = 0.023). ..272

CHAPTER - I

INTRODUCTION.

1.1. GENERAL:

In an effort to save weight, while still retaining high strength capabilities, many contemporary structural systems are designed with lower margins of safety than their predecessors. The criterion of minimum weight design is particularly prevalent in the design of aircraft, missile, and space craft vehicles. One obvious means of obtaining a high strength, minimum weight design is the use of light, thin-walled structural members of high strength alloys. For intricate structures such as space-crafts, beams of standard cross section may not be the most efficient or convenient structural members to use. Thin-walled beams of open section are frequently employed for their structural efficiency. With the improvement of extrusion methods in metal forming, beams of different shapes of cross sections can be formed to order. Occasions often arise when uniform doubly symmetric cross sections are more convenient to use. Examples of such structural members that have gained great favour as stiffeners in aerospace design are the I,Z, Channel and angle sections.

Although no attempt has been made in the previous paragraph to regorously define a thin-walled beam, it is necessary to do so in order that one fully understands its meaning when used in ensuing discussion. A rectangular beam as a structural member is characterized by having two dimensions, the width and depth of the cross section of comparable size but small in comparision with the third dimension, the length. A thin-walled beam, on the other hand, is characterized by its three dimensions being of different orders of magnitude. The thickness of the beam is small compared to the characterestic dimensions of the cross section, and the cross sectional dimensions are small compared to the length of the beam.

It has long beam known that a beam with nonsymmetrical cross section under loads will, in general, not only deflects but also will also twist. Only under special loading along the flexure axis, a line joining the shear centers, will the beam deflect without twist. The concept of shear center is well known and is discussed in text books. Essentially, it is a point through which the resultant of the shear forces of the cross section passes. If the loading does not pass through the shear center, a torque is generated by the loading and the resultant of the reactions from the section. Such a torque will cause the twisting of the beam. When a thin-walled beam is subjected to dynamic excitation, the inertial loading due to acceleration of the beam itself has to be taken into account. The resultant of such loading may be considered to pass through the centroid of the section. Unless the shear center of the section coincides with its centroid, both bending and torsional vibrations will result. Due to the low torsional rigidity of thin-walled open section beams, the problem of torsional vibrations and stability is of primary interest.

1.2. BRIEF REVIEW OF RELEVANT LITERATURE:

Extensive research has been conducted in the field of thin-walled structural members which has been well documented in the literature, and detailed bibliographies are already available. Therefore, only a brief survey of the development of the existing literature directly related to the present investigation will be included here.

1.2.1. ELASTIC STABILITY:

Since the eighteenth century investigation of column instability by Euler, a great wealth of information has been documented concerning the nature of instability. For instance, the instability of columns, beam-columns, plane frames, trusses, plates, and shells have been the objects of many research efforts. Although the indifidual investigations are too numerous to cite, several texts have appeared that provide excellent anthologies for these investigations.

Derivation of the fundamental theory of strength and stability of thin-walled members was performed by Goodier, Timoshenko, Vlasov and others. Timoshenko (95) initiated the concept of non-uniform torsion when he considered warping of the cross sections of a symmetrical I-beam subjected to torsional moment. Wagner (110) generalized the Timoshenko torsion theory. Goodier (36,37) published a series of studies in which he simplified and proved some of the assumptions proposed by earlier investigators. Theories of lateral stability and flexural-torsional stability of uniform thin-walled beams, upto 1945, were unified by Timoshenko (98). Vlasov's (107) extensive investigations of thin-walled elastic members were published in book form in 1940. A new edition containing comprehensive study of equilibrium, stability, and Vibration of thin-walled members of arbitrary cross sections was published in Russian in 1958 and translated into English in 1961.

Two other classical text books dealing with the stability of members were published by Bleich (13) in 1952 and Timoshenko and Gere (99) in 1961. Most recent is Ziegler's monograph (114), in 1968, on structural stability in which he emphasizes the conceptual aspects of the more recent developments of stability theory. Surveys of the theory of thin-walled members, which include numerous references, were performed by Nowisinki (87) in 1959, Panovko (89) in 1957 and Yi-Yuan, Yu (113) in 1971. A survey of literature on the lateral instability of beams was made in 1960 by Lee (73). The effect of axial stresses, arising from combined bending and torsion of thin-walled beams, on the torsional regidity of the beam was investigated by Goodier (38) in 1951 and Engel (29) in 1953.

In 1944, Goodier and Barton extended Timoshenko's theory of non-uniform torsion of an I-beam to include not only the bending of the flanges in their own planes but also considered the effect of web deformation on the torsion of the beam (~~15~~39). Further investigation of this effect including experimental work was performed by several researchers. The Goodier-Barton effect was found to be of significant importance for the case of plate girders whose cross sections were such that the ratio of the flange thickness to the web thickness was large or if the length of the web was much larger than the length of the flange (35,71).

Gregory (42) in 1961, proposed a theory which considered a non-linear longitudinal stress system in members subjected to large elastic torsional displacements. Gregory's theory was developed by Black (11,12) in 1965 and in 1967, in a theoretical and experimental study of monosymmetric thin-walled beams subjected to bending and torsion. Approximate solutions of a modified non-linear equation were compared with the experimental results and also with the theories of Timoshenko (98) and Goodier (38). A continuous effort has also been made to close the gap between structural theory and engineering codes of practice (5,18,12). Recent research studies of interest to designs and research workers are presented in a collection of papers, published in 1967, on the stability and strength of thin-walled structural members and frames (16).

The influence of second order effects such as distortion of the column cross section, large displacements, shear deformation, residual stress and initial deflections on the behaviour of biaxially loaded columns ~~is~~ was evaluated by Culver (22) in 1965. Numerical calculations, including these second order effects, indicated that problems exist for which these effects are considerable. Second order effects influencing biaxially loaded columns were discussed by Goodier (40) and Heilig (44) and these effects included cross sectional distortion due to torsion and shear deformations.

Tapered thin-walled beams are of interest in optimum design. Gere and Carter (33) obtained the critical buckling loads for tapered columns. A finite element formulation using Gelerkin's method for the buckling problem of tapered members was presented by Morrel and Lee (82). The elastic stability of axially loaded tapered columns has been studied analytically by several investigators (27,84). The problem of torsional buckling of axially loaded tapered columns of wide-flanged cross section has been recently studied analytically by Culver and Preg (23), using finite-difference method. In addition, the differential equations for the general case of tapered wide-flanged beam-columns have been derived using the Vlasov's method (107) for uniform beams. The determination of the initial yield load for tapered beam-columns has also been investigated (30). An experimental investigation of the elastic stability of tapered beam-columns has been reported (15). Lee (74) presented an analysis of non-uniform torsion of tapered I-beams in 1956, the taper being only of a restrictive type.

All the above investigations and a host of others treat the torsional or flexural - torsional buckling problems from a purely mathematical approach. Such an approach includes the solution of a trio of coupled differential equations of equilibrium (these equations may be uncoupled under some instances) for columns of various cross sections, loadings and boundary conditions. This approach provides one with exact solutions (mathematically speaking) for a given problem. One shortcoming of such an approach is that due to the complex nature of the equilibrium equations such mathematical difficulties as non-uniform members, complex loadings, or arbitrary boundary conditions can not be easily handled.

To complement the known exact solutions, attempts have been made to obtain approximate solutions to the more difficult (again, mathematically speaking) problems. The technique used to obtain the approximate results is the method of finite or discrete element technique. Many of the early advances in the finite element method were presented in technical journals, but recently texts by Przemieniecki (93) and Zienkiewicz (115), have appeared that summarized various investigations utilizing this modern technique. These texts cover such varied topics as plane stress, plane strain, axisymmetric stress analysis, three dimensional stress analysis, bending of beams, plates and shells, vibrations of elastic systems, and structural stability.

Using the finite-element technique, Krajcinovic (65) developed a formulation for thin-walled members based on the use of hyperbolic functions to express the twist. These functions, which are the solution to the exact differential equation for twist, lead to complicated stiffness expressions in torsional and warping constants. It does not include the effects of instabilities due to torques. Hence, its applicability to general frame instability is limited. Kabaila and Fraeijsde Venbeke(46) formulated a finite-element model that considers only axial forces in the stability analysis. The formulation is only applicable to solid beams where the shear center coincides with the center of gravity. It neglects warping rigidity, which is of major importance in the analysis of thin-walled members(98). A linear formulation was used to express the twist, as was done earlier, by Przemieniecki (73). The finite-element method has been shown, by Pardoen (90), Barsoum (6, 8) and Barsoum and Gallangher (7) to be completely general in that it provides one with a means of solving problems involving arbitrary loading and boundary conditions. Although, only an approximate method, the finite-element method has provided results that are sufficiently accurate for engineering purposes.

1.2.2. VIBRATIONS AND WAVE-PROPAGATION:

For the past three decades mechanical vibrations have been recognized as a major factor in the design of air craft, marine and machine structures. Mechanical vibrations produce increased stress, energy loss and noise that should be considered in the design stages if these undesirable effects are to be avoided, or kept to a minimum. This is essentially true in the area where the total mass of the system is to be held to a minimum. Vibratory motion can produce very disastrous results as in the case of either the Tacoma narrows bridge which fell because the wind excited it at a natural frequency, or the ill-fated Electra I Commercial air craft that encountered severe engine vibration which required major modification of air craft.

The important point to be noted is that too often vibrations are investigated after, instead of before, the failure has occured.

Several investigators have been concerned with the vibration of beams and the purpose herein is to review some of the relevant contributions in this area. The most desirable technique for analyzing vibratory motion is the regorous mathematical solution obtained from a formal solution of the differential equations describing the motion. Timoshenko, and Young (100) investigated the coupled torsional and transverse vibrations of a simply supported beam having a constant channel cross section. He considered only the simply supported beam and by assuming a product form solution were was able to obtain an algebraic frequency equation. This technique is limited to only those cases in which it is possible to assume a solution for the mode shape that satisfies the physical constraints of the beam. Gere (31) studied the torsional vibration of beams with doubly-symmetric cross section for which the shear center and centroid coincide and analyzed the effect of warping on the frequencies of torsional vibration and the shapes of the normal modes of vibration for bars of single span with various end conditions. Gere and Lin (34) generalized the theory of vibrations of thin-walled beams of arbitrary open section.

The above cited references presented classical mathematical solutions for the beam vibration problems. Wherever possible the use of these formal mathematical solutions is highly recommended because they are the simplest and most direct methods of predicting vibratory characterestics. However, it should be noted that each of these formal solutions has very definite limitations because they have been obtained for a specific type of beam and are not applicable to the general case. Since there had not been developed a rigorous mathematical technique that will solve all types of beam vibration problems, it was only natural that various approximate techniques have been developed to fill in the gaps left in the formal solutions. One of the most powerful techniques developed was the Rayleigh-Ritz method which is an energy principle that in the absence of frictional losses, the total vibratory energy of a vibrating body must continuously change from all strain energy and no kinetic energy to all kinetic energy and no strain energy, and the frequency of change must be a natural frequency.

The first step in the application of the Raleigh-Ritz method is to assume a possible model shape of the beam corresponding to the lowest frequency. Then it will be possible to calculate the maximum strain energy in the beam. By considering that the assumed mode shape is periodic in time the maximum kinetic energy can be obtained. When the two energies are equated, it is possible to solve for the frequency. Succeeding possible mode shapes must be assumed until the lowest calculated frequency is obtained. This technique converges only to the lowest natural frequency of the system. The higher natural frequencies can be obtained only by using the orthogonality property that exists between the mode shapes. A complete discussion of the Raleigh-Ritz technique is presented by Temple and Beckley (96).

Garland (41) used the Rayleigh-Ritz method to investigate the coupled torsional and transverse vibration of cantilever beams having constant channel cross section. He was able to observe that for any one transverse mode of vibration there will be two torsional modes and that the coupled natural frequency can be expressed as functions of the uncoupled transverse and uncoupled torsional frequencies. Timoshenko (100) was also able to make this observation for a simply supported channel cross-section. Garland was able to obtain a remarkable degree of correlation between the predicted and the experimentally measured results. Because he was dealing with only the lowest natural frequencies, he was not in requirement of the use of the orthogonality condition that would be necessary for obtaining the higher natural frequencies.

Bennett (9) developed an improved matrix technique for investigating the vibratory characterestics of a beam having a plane of symmetry perpendicular to the plane of transverse vibration. For a beam having a non-collinear longitudinal mass and shear center axes, there will be a coupling between the transverse and torsional vibrations. The coupling is produced when the reversed effective force caused by the transverse vibration does not act through the shear center of the cross-section. To date there has not been developed a rigorous mathematical solution for all possible variations in cross section, loading conditions and methods of support. Several authors have solved the equations by imposing specific limitations on the method of support or on the variation of the cross section. Some researchers have used an energy method or an iterative method to approximate solutions where the formal solution does not exist. These approximate methods have a tendency to become very tedious. The technique of investigating the higher natural frequencies introduces complexities that are difficult to understand physically. The matrix method proposed by Bennett (9) is valid for any loading conditions or method of support. In his work, three different types of beam vibrations are considered, coupled torsional and transverse, transverse alone and torsional alone. The governing differential equations were solved approximately by using a digital computer and results obtained are observed to be within the range of engineering accuracy.

Another approximate but more elegant technique is the finite-element technique which provides one with solutions for any general set of boundary conditions and the variation in the cross section. This technique has been successfully used by Mei (77,78) for the solution of the coupled bending-torsion vibrations of thin-walled beams of open section and non-linear flexural vibrations of rectangular beams. Pardoen (90) and Barsoum (6) presented satisfactory solutions for the vibration and dynamic stability problems of thin-walled beams of open section utilizing the finite-element method. Although the finite-element technique has been used to predict the natural frequencies and mode shapes of beams, the method has yet to be extended to consider the torsional vibrations and stability of thin-walled beams of open section resting on continuous elastic foundation.

Stress wave propagation in elastic solid media have been subjected to analysis since the early investigations of poisson (92). Recent developments have been motivated by the ever increasing need for information concerning the response of structures to high dynamic loads. The beam as a fundamental element of structures, received the first attention of investigators in the field. The early work of Pochhammer (91) and Chree (17) on the circular cylindrical bar with traction-free surface was re-examined in the early 1940's but progress was slow on account of highly intricate transcendental frequency equations resulting from dispersion due to the presence of boundaries. The first three modes of longitudinal and flexural wave transmission were not known until found by Davies (24) in 1948 and Abramson(1) in 1957.

The complexity of the exact analysis even for simple geometry of a circular cylindrical bar, emphasized the need for physically satisfactory approximate theories. To satisfy engineering requirements, these theories should be good for short wave lengths which occur in problems of steep transients or high frequency oscillations in bars. The elementary classical theories of Navier for longitudinal vibrations, Bernoulli-Euler for flexural vibrations and Coulomb for torsional oscillations were reviewed and with the exception of the latter, were found to lead to physically impossible results (72). As a consequence, emphasis was placed on developing more accurate approximate theories for longitudinal and flexural vibrations.

Although Timoshenko (101) in 1921 proposed a theory for flexural oscillations which included the effects of shear deformation and rotary inertia, it was not until the last decade that the Timoshenko theory was really put to experimental and analytical tests. During this period, in addition to a lot of allied literature on exact theories of plates, and over a dozen of books, monographs and surveys, not less than fifty papers appeared dealing with approximate theories. These papers included new theories; their mutual comparison, comparision with the known information from exact theories and experiment. The Timoshenko theory for flexural waves and the Mindlin-Herrmann theory (81) for longitudinal waves were found most satisfactory. The rest of literature with the propagation of pulses is based on these theories. Brief details have been previously summarized by Kolsky (67), Abrahmson, Plass and Ripperger(2), Green (41), and more recently by Redwood (89) and Miklowitz(80

However, comparable torsional oscillation analysis was virtually neglected and not more than four to five papers on the topic have been published. The reason is the fact that Coulomb classical theory gives the same first-mode results as the exact theory. The available information is almost limited to the circular cylindrical bar. Thus, there exists a lack of satisfactory approximate and exact theories for torsional wave propagation in non-circular bars, especially these used in structural applications. Very often thin-walled beams of open section are used as structural members in light weight aircraft and building construction. These members usually fail under torsion or combined bending torsion because of their low torsionaly rigidity which makes them susceptible to torsional buckling. A self-contained and comprehensive account of bending and torsion of thin-walled beams of open section was given in a paper published by Timoshenko (98) in 1945. As structural members may be subjected to resonant vibrations under dynamic loads, it is necessary to study their torsional properties in order to understand their response to torsional excitation.

The inadequacy of a Saint-Venant elementary torsion theory for short wave lengths was hinted at by Love (76), who suggested a correction for the longitudinal inertia associated with torsional deflection. However, both the elementary theory and Love's approximation have the same defects as do their counterparts in longitudinal wave-propagation theory. The dynamic equation used by Gere (32) in his torsion analysis was essentially that previously derived by Timoshenko(98) and he studied the effect of warping of the cross-section on the frequencies of vibration. These equations ~~are called~~ Constitute the Timoshenko Torsion theory in the sequel and are found to lead to physically absurd results for short wave length waves.

To present a much needed practical engineering theory, a strength of materials theory was is derived and analyzed by Aggarwal (3) in his thesis, including the effects of shear deformation, longitudinal inertia and warping of the cross-section. At high frequencies and short wave lengths a new mode of the wave transmission is added. This arises from the coupled interaction of the torsional deformation and bending effects of shear deformation and longitudinal inertia. The Aggarwal's theory lead to theoretically satisfactory results for the first mode of transmission over a wave length spectrum which included moderately short wave lengths, and agrees with previous approximations for large wave lengths. The group velocity for the second mode is shown to increase monotonically from zero for the longest waves to the bar velocity for very short wave lengths, which is in agreement in form with the higher modes of the exact theory for circular cylindrical bars (88, 23). In many respects the analysis of Aggarwal's theory proves to be analogous to that of Timoshenko's flexural theory (101).

The transient response arising from a step torque applied impulsively at the end of a semi-infinite I-beam is analyzed by Aggarwal (3) and the non-dimensional equations ~~are~~ are solved using Laplace transforms and a closed form solution in integral form is obtained. For the sake of comparison, he solved the same impulsively applied step torque problem according to the Timoshenko torsion theory. He also analyzed the problem of free and forced vibrations of I-beams according to his theory which includes the effects of longitudinal inertia and shear deformation. He noticed a completely new spectrum of natural frequencies at higher frequencies due to the interaction between torsion, shear deformation and longitudinal inertia effects The frequency equations and expressions for model functions are derived for a number of cases but he limited the discussion regarding the existence of the second frequency spectrum only to the case of the simply supported beam because of the highly transcendental nature of the frequency equations which further include the parameters of warping, shear and longitudinal inertia. The frequencies obtained according to his theory are we compared with those previously obtained by Gere (32) who used the Timoshenko torsion equation. The shear effect is shown to result in a decrease of beam stiffness and corresponding decrease of natural frequencies. Though, the decrease is relatively small compared to the increase due to warping; the influence of shear deformation is observed to be considerable at higher frequencies. Further, Aggarwal (3) established an Orthogenality relation for the principal modes of vibration and treated the problem of forced vibrations under very general loa

Where as Aggarwal's contribution was limited to an improvement of the previous theories of uncoupled torsional vibrations, Tso's contribution (104) was in the field of coupled torsional and bending vibrations of thin-walled beams of open section. In his thesis, Tso (104) derived a higher order theo including the effect of shear strain induced by bending and warping of the beam. He compared the spectrum curves of the highe order theory with those from the elementary theory for various boundary conditions for a special family of non-symmetric sections. He performed an experiment on two specimens to determi their natural frequencies at different beam lengths and compared the experimental results with those predicted from the two theories. He has concluded that when the beam is long, the elementary theory is adequate to predict the natural frequencies predominantly torsional predominantly bending

for torsion predomenant modes. For bending predominant modes, the higher order theory should be used. The higher order theory derived by Tso (104) serves also as a guided for the range of validity of the elementary theory. In the experimental observations, he found certain non-linear behaviour of the thin-walled beam. Under special circumstances, when the beam is excited at resonance at a higher mode, he observed a tendency for the beam to shift from the higher resonant mode to vibrate at its fundamental mode, resulting in a higher order subhormonic oscillation. Hence he made an analysis to show the possibility of such a behaviour if the inherently non-linear governing equations for coupled torsional and bending vibrations are used.

Recently In 1967, Aggarwal and Cranch (4) published a paper as an extension to the work of Aggarwal (3), by including an analysis for the coupled bending-torsional vibrations of a channel beam. The equations governing the motion of the channel beam are derived using Hamilton's principle and include the effects of warping, longitudinal inertia and shear deformation. These equations explicitly resemble those derived by Tso (104) for the more general case of mono-symmetric thin-walled beam of open cross section. However, the approach of Aggarwal and Cranch seems to be different from that of Tso. Whereas Tso, analyzed the vibrations of a monosymmetric thin-walled beam, torsional wave analysis ~~is~~ was made by Aggarwal and Cranch for the case of an I-beam and a channel beam.

A more general theory of vibrations of cylindrical tubes which includes the secondary effects such as transverse shear, longitudinal inertia and shear lag was presented by Krishnamurthy and Joga Rao (70). They also brought out the analogy between the flexural and torsional vibrations of doubly symmetric tubes. In Part IV of their theory (70), results for simply supported open tube of doubly symmetric I - section were presented. The other boundary conditions were not analyzed.

1.3. AIM AND SCOPE OF THE PRESENT INVESTIGATION:

In the above investigations (34, 78, 80) on the torsional vibrations of thin-walled beams of open section including the second order effects such as longitudinal inertia and shear deformation, only regorous mathematical solutions are attempted. This approach actually limited their solutions only to simple end conditions such as a simply supported beam. Stating that, the frequency equations are highly transcendental in nature, Aggarwal (3) did not attempt the solutions for boundary conditions other than the simply supported ends. However, with the advent of high speed digital computers, it is not too difficult to obtain the solutions for these transcendental frequency equations.

The present thesis aims at developing exact and approximate methods of analysis to tackle various boundary conditions without much difficulty. An attempt has been made, to extend the previous discussions on torsional vibrations and stability analysis of thin-walled beams of open section, to include the effects of axial compressive load, continuous elastic foundation, longitudinal inertia and shear deformation by making use of exact and approximate methods of analysis. A non-linear analysis is also made to study the influence of large torsional amplitude on the non-linear period of vibration. Further, the effects of axial compressive load and continuous elastic foundation on non-linear torsional behaviour of thin-walled beams of open section are also investigated.

In particular, Chapter II deals with the analysis of torsional vibrations and stability of lengthy uniform thin-walled beams of open section resting on continuous elastic foundation and subjected to a time-invarient axial compressive load by means of exact and approximate methods. A finite-element formulation for the same problem which is useful both for uniform and non-uniform beams is presented in Chapter III. The comparison between the results from the exact analysis and approximate finite element method is shown to be excellent even for a coarse sub-division of the beam.

In Chapter IV, an exact analysis is presented for free torsional vibrations of short uniform thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. Expressions for orthogonality and normalizing conditions for the principal normal modes which are useful in solving forced vibration problems and which include both the angle of twist and warping angle are obtained for both the general case and for beams with various simple end conditions. To fescilitate use by the designers, extensive design data pertaining to wide-flanged I-beams with various end conditions are is presented. Also, approximate frequency equations for clamped and clamped-simply supported beams are derived making use of the Galerkin technique. A finite element formulation of the problem is presented in Chapter V. New stiffness and mass matrices are presented which included the effects of longitudinal inertia and shear deformation. The results obtained by the finite element method are in good agreement with the exact ones.

An analysis for the forced torsional vibrations of thin-walled beams of open section including the effects of longitudinal inertia, shear deformation and viscous damping is given in Chapter VI. Chapter VII deals with the problem of torsional wave propagation in orthotropic thin-walled beams of open section including the effects of longitudinal inertia and shear deformation.

In Chapter VIII, the problem of torsional vibrations and stability of short uniform thin-walled beams resting on continuous elastic foundation and subjected to an axial static compressive load including the effects of longitudinal inertia and shear deformation is analyzed by means of an exact method. Approximate expressions for the frequency and buckling load are derived for clamped and clamped-simply supported beams utilizing Galerkin's technique. A finite-element solution of the same problem is presented in Chapter IX.

A non-linear analysis for the torsional stability of thin-walled beams of open section at large amplitudes is presented in Chapter X. In Chapter XI, the effects of axial time-invariant compressive load and elastic foundation on the non-linear torsional vibrations and stability are analyzed. In Chapter XII, salient conclusions are arrived at, bringing out the practical significance of the problems solved. Also the scope for further investigation is discussed.

Available reprints of the papers published on part of the work presented in this thesis are enclosed at the end for ready reference. The rest of the material is accepted for publication in reputed Journals and is awaiting publication.

CHAPTER - II

TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS ON ELASTIC FOUNDATION - EXACT AND APPROXIMATE ANALYTICAL SOLUTIONS*

2.1 INTRODUCTION:

Static and dynamic analysis of beams on elastic foundation occupies a prominant place in contemporary structural mechanics. The vibrations and buckling of continuously supported finite and infinite beams resting on elastic foundation has an application in the design of highway pavements, aircraft runways and in the use of metal rails for rail road tracks. A very large number of studies have been devoted to this subject, and valuable practical methods for the analysis of beams on elastic foundation have been worked out.

Regarding the static analysis of beams on elastic foundation Hatenyi's book (43) ~~is rather a~~ classic giving the complete development of the beams supported on elastic foundation. A later development of the theory is beautifully presented by Vlasov and Leovitiv (108) in their book on ''beams, plates, and shells on elastic foundation'' with improved models of elastic foundation. Since the actual response at the interface depends on the material of the foundation and is usually very difficult to determine, various foundation models were proposed to approximate the real foundation behavior among which Winkler's constant modulus foundation is widely used because of its simplicity. A discussion of various foundation models is presented by Kerr (65).

---

* Part of the results from this chapter were published by the author and A.A.Satyam in February 1975 issue of AIAA Journal, see Ref. 47.

The effect of shear flexibility is included in the analysis of beams on elastic foundation by Ractliffes (74). Biot (10) treated the bending of an infinite beam on elastic foundation and Conway and Farnham (19) analyzed the bending of a finite beam in bonded and unbonded contact with an elastic foundation. Recently Niyogi (86) presented an approximate analysis of axially constrained beam on elastic foundation and Murthy (83) solved the problem of buckling of continuously supported beams. The problem of buckling of thin-walled beams of open section such as I-beams, channel sections etc., with continuous elastic supports has been treated by Timoshenko and Gere (77) in their book on ''Theory of elastic stability''. By using the finite element method, Pardoen (90) analyzed the buckling of thin-walled beams of open section resting on continuous elastic supports subjected to an axial load.

On the dynamics side of beams on elastic foundation, Kenney (66) analyzed the steady state flexural vibrations of beams on elastic foundation for a moving load including the effect of viscous damping. Crandall (20) analyzed the flexural vibrations of a beam on elastic foundation including the effects of rotary inertia and shear deformation. Tseitlin (103) determined the effects of shear deformation and of rotary inertia in flexural vibrations on beams on elastic foundation. Lloyd and Miklowitz (75) presented an analysis for the flexural wave propagation of beams and plates on an elastic foundation.

While there exists a good number of investigations on flexural vibrations of rectangular beams or plates on elastic foundation, the literature on the torsional vibrations of beams on elastic foundation is rather scarce. To the best of authors knowledge the effects of a time-invarient axial compressive load and of elastic foundation on the torsional frequency and buckling loads of thin-walled beams of open section are not being analyzed anywhere in the available literature. To this end, the present chapter deals with the exact and approximate analytical solutions of the effects of a time-invariant axial compressive load and of elastic foundation on the torsional frequency and buckling loads of lengthy thin-walled beams of open section.

2.2. BASIC ASSUMPTIONS:

The problem investigated in this chapter is restricted to the following assumptions:

a) The thin-walled beam has uniform open cross sections along its length.

b) Strains are assumed to remain within the elastic limit. The curvature and twist of the beam are considered to be small. In particular, the deformations are small compared with the cross-sectional dimensions of the beam in the linearized problem.

c) The beam is fabricated from material which is homogeneous and isotropic and which obeys Hooke's law (a linearly elastic material).

d) The centroid and shear center of the cross section coincide.

e) Shearing strains of the middle surface due to shear and warping effects, and axial strains of the beam due to longitudinal load components are considered to be negligibly small (the beam is undergoing inextensional motions).

(f) Longitudinal inertia effects are considered to be negligibly small. Conditions (e) and (f) are referred to as the Timoshenko Torsion theory.

(g) Distortion of the cross sections in their own planes is not considered, however, warping of the sections is permitted. Distortion of the sections would be of significance for built-up girders or if the cross section is very deep or very wide.

(h) No internal or external damping forces are considered.

2.3 DERIVATION OF BASIC DIFFERENTIAL EQUATION:

As the cross sectional dimensions are assumed to be small compared to the length of the beam, the second order effects such as longitudinal inertia and shear deformation can be treated as negligible.

In this section, based on Timoshenko torsion theory (98), the governing differential equation of free motion of a doubly symmetric thin-walled beam on elastic foundation subjected to a time-invariant axial compressive load is derived utilizing Hamilton's principle. The method has the advantage of generating the natural boundary conditions which shall be discussed in section 2.4.

Hamilton's principle (8 76), states that for dynamical process:

$$\delta \int_{t_0}^{t_1} (T_k - U + W) \, dt = 0 \qquad (2.1)$$

where (T - U + W) is the Lagrangian function, T_k the kinetic energy of the strained bar, U the total strain energy, W the potential energy of the external force, and t_o, t_1 are two fixed instants.

Fig.1.1 shows a differential element of length dz of a wide-flanged I-beam undergoing torsion. According to Saint Venant, the cross-sections are assumed to rotate about the centroid-shear center 'O' giving rise to a torsional couple,

$$T_s = GC_s \frac{\partial \phi}{\partial z} \qquad (2.2a)$$

where G is the shear modulus, C_s the torsion constant for the cross section, and ϕ (z, t) the angle of twist.

The torsion constant for an I-section is given by

$$C_s = (2bt_f^3 + ht_w^3)/3 \qquad (2.2b)$$

where b is the width of the flanges, h the height between the centerlines of the flanges, t_f the thickness of the flanges, and t_w the thickness of the web.

The strain energy U_1 at any instant t in the beam of length L due to Saint Venant torsion is

$$U_1 = \frac{1}{2} \int_0^L GC_s \left(\frac{\partial \phi}{\partial z} \right)^2 dz \qquad (2.2c)$$

Accompanying the rotation is a warping of the section which is assumed constant in each piece of the cross section hava moment M. The x-displacement of the top flange centerline, u

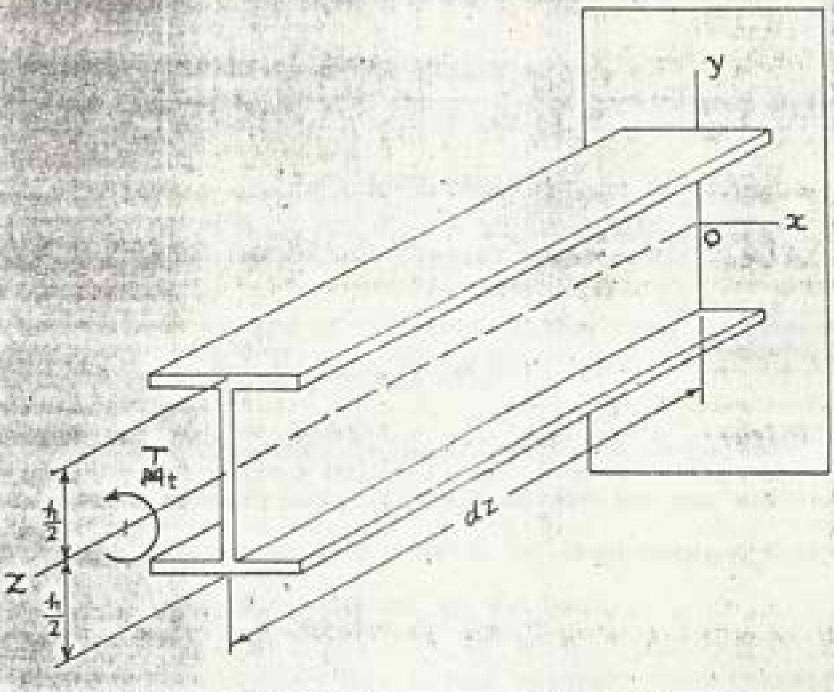


FIG. 2·1 – DIFFERENTIAL ELEMENT OF A WIDE-FLANGED I-BEAM is given by

$$u = (h/2)\,\phi \qquad (2.2d)$$

and hence the moment M in the top flange is given by

$$M = EI_f \frac{\partial^2 u}{\partial z^2} = EI_f \frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} \qquad (2.2e)$$

where E is the Young's modulus, I_f the moment of inertia of each flange area about y-axis.

It can be easily observed that the moment M in the top flange and -M in the bottom flange cancel so that no net moment M_y exists in the cross section.

The shear force Q due to the bending of the flanges is given by

$$Q = \frac{\partial M}{\partial z} = EI_f \frac{h}{2} \frac{\partial^3 \phi}{\partial z^3} \qquad (2.2f)$$

The equal and opposite shear forces Q, a distance h apart in the top and bottom flanges, give rise to a torque due to warping, T_w, given by

$$T_w = -\,Qh = -\,EI_f \frac{h^2}{2} \frac{\partial^3 \phi}{\partial z^3} = -\,EC_w \frac{\partial^3 \phi}{\partial z^3} \qquad (2.2g)$$

where $C_w = I_f\, h^2/2$ is the warping constant for an I-section (32).

The total torque, T_t, on the cross section is given by

$$T_t = T_s + T_w = GC_s \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} \qquad (2.2h)$$

If U_2 is the strain energy of the two flanges due to warping, then

$$U_2 = \frac{1}{2} \int_0^L 2\, EI_f \left(\frac{\partial^2 u}{\partial z^2}\right)^2 dz = \frac{1}{2} \int_0^L EC_w \left(\frac{\partial^2 \phi}{\partial z^2}\right)^2 dz \qquad (2.2i)$$

The strain energy U_3 due to the Winkler type elastic foundation, is given by

$$U_3 = \frac{1}{2} \int_0^L K_t(\phi)^2\, dz \qquad (2.2j)$$

Hence, the total strain energy U, at any instant t becomes

$$U = U_1 + U_2 + U_3 = \frac{1}{2} \int_0^L \left[GC_s \left(\frac{\partial \phi}{\partial z}\right)^2 + EC_w \left(\frac{\partial^2 \phi}{\partial z^2}\right)^2 + K_t(\phi)^2 \right] dz \qquad (2.2)$$

The kinetic energy of rotation of the cross section at the corresponding instant is given as:

$$T = \frac{1}{2} \int_0^L \rho I_p \left(\frac{\partial \phi}{\partial t}\right)^2 dz \qquad (2.3)$$

where I_p is the polar moment of inertia of the cross section and ρ the mass density of the material of the beam.

The potential energy due to the external time-invariant axial compressive load, P, acting at the centroid of the cross section at the corresponding instant is given by

$$W = \frac{1}{2} \int_0^L \frac{PI_p}{A} \left(\frac{\partial \phi}{\partial z}\right)^2 dz \qquad (2.4)$$

where A is the area of the cross section.

Substituting for T_t U and W from equations (2.2) to (2.4) respectively in equation (2.1), taking the variations of the integrand, and integrating the first term by parts with respect to t and the next four terms with respect to z, one obtains:

$$\int_{t_0}^{t_1}\int_{0}^{L}\left\{\left|(GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} - EC_w\frac{\partial^4\phi}{\partial z^4} - K_t\phi - \rho I_p\frac{\partial^2\phi}{\partial t^2}\right|\delta\phi\right\} dz\,dt$$

$$+ \int_{0}^{L}\rho I_p\frac{\partial\phi}{\partial t}\,\delta\phi\,\Bigg|_{t_0}^{t_1} dz - \int_{t_0}^{t_1} EC_w\frac{\partial^2\phi}{\partial z^2}\,\delta\,(\frac{\partial\phi}{\partial z})\Bigg|_{0}^{L} dt$$

$$- \int_{t_0}^{t_1}\left\{(GC_s - \frac{PI_p}{A})\frac{\partial\phi}{\partial z} - EC_w\frac{\partial^3\phi}{\partial z^3}\right\}\delta\phi\,\Bigg|_{0}^{L} dt = 0 \qquad (2.5)$$

Assuming that the values of ϕ are given at the two fixed instants, the second integral vanishes. If the boundary conditions are such that the third and the fourth integrals also vanish, then the associated differential equation of motion is given by:

$$(GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} - EC_w\frac{\partial^4\phi}{\partial z^4} - K_t\,\phi - \rho I_p\frac{\partial^2\phi}{\partial t^2} = 0 \qquad (2.6)$$

2.4 (a) NATURAL BOUNDARY CONDITIONS:

In deriving the basic differential equation of motion (2.6) from (2.5) it was assumed that the expressions

$$EC_w\frac{\partial^2\phi}{\partial z^2}\quad\delta(\frac{\partial\phi}{\partial z}) \qquad ($$

and

$$\left[(GC_s - \frac{PI_p}{A})\frac{\partial\phi}{\partial z} - EC_w\frac{\partial^3\phi}{\partial z^3}\right]\delta\phi$$

vanish at the ends z = 0 and z = L. These conditions are satisfied if at the two ends

$$\frac{\partial^2\phi}{\partial z^2}\,\delta\left(\frac{\partial\phi}{\partial z}\right) = 0, \qquad (2.7)$$

and

$$\left[(GC_s - \frac{PI_p}{A})\frac{\partial\phi}{\partial z} - EC_w\frac{\partial^3\phi}{\partial z^3}\right]\delta\phi = 0 \qquad (2.8)$$

Equation (2.7) and (2.8) give the natural boundary conditions for the finite bar, and are satisfied if the end conditions are taken as

(1) $\phi = 0$ and $\frac{\partial^2\phi}{\partial z^2} = 0$ (2.9)

These conditions imply restraint against rotation but not against warping; that is, the end of the bar does not rotate but is free to warp. This is the case of a ''Simple Support''.

(2) $\phi = 0$ and $\frac{\partial\phi}{\partial z} = 0$ (2.10)

These conditions imply restraint not only against rotation but also against any warping of the end cross section. This means that the end of the bar is built-in rigidly so that no deformation of the end cross section can take place. These conditions define a ''Fixed Support''.

(3) $\frac{\partial^2\phi}{\partial z^2} = 0$ and $(GC_s - \frac{PI_p}{A})\frac{\partial\phi}{\partial z} - EC_w\frac{\partial^3\phi}{\partial z^3} = 0$ (2.11)

These conditions imply no restraint of any kind at the end of the bar. This requires that the bending moment in the flange ends and torque acting on the end cross section must be zero. These conditions correspond to a ''free end''.

(4) $$\frac{\partial \phi}{\partial z} = 0 \text{ and } (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} = 0$$

or equivalently

$$\frac{\partial \phi}{\partial z} = 0 \text{ and } \frac{\partial^3 \phi}{\partial z^3} = 0 \qquad (2.12)$$

The latter conditions imply no warping and zero shear forces in the end flanges.

These conditions are useful for finding symmetric modes of vibration in simply supported, fixed-fixed and free-free beams.

(b) TIME-DEPENDENT BOUNDARY CONDITIONS:

The homogeneous boundary conditions discussed above, give the free vibrations of bars. For forced vibrations produced by the motion of boundaries, appropriate time dependent end conditions are given by prescribing at each end one member of each of the products:

$$EC_w \frac{\partial^2 \phi}{\partial z^2} \bar{\delta}(\frac{\partial \phi}{\partial z}) \text{ and } \left| (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} - EC_w \frac{\partial^3 \phi}{\partial z^3} \right| \bar{\delta}\phi$$

or equivalently of:

$$M \bar{\delta}(\frac{\partial \phi}{\partial z}) \text{ and } T_t \bar{\delta}\phi.$$

Of the many conditions thus obtained, the following are of more theoretical interest:

1. Twisting moment T_t prescribed, flange bending moment $M = 0$ or $\frac{\partial \phi}{\partial z} = 0$,
2. ϕ or $\frac{\partial \phi}{\partial t}$ prescribed, flange bending moment $M = 0$ or $\frac{\partial \phi}{\partial z} = 0$,
3. Flange bending moment M prescribed, twisting moment $T_t = 0$ or $\phi = 0$,
4. $\frac{\partial \phi}{\partial z}$ or $\frac{\partial^2 \phi}{\partial z \partial t}$ prescribed, twisting moment $T_t = 0$ or $\phi = 0$.

In the case of semi-infinite beams, conditions need be prescribed at one end since all physical quantities at any instant are zero at the far end.

2.5 ANALYSIS OF VARIOUS TERMS:

i) If $K_t = P = 0$ and $C_w = 0$, Eq.(2.6) reduces to

$$GC_s \frac{\partial^2 \phi}{\partial z^2} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.13}$$

This equation represents Saint Venant's torsion theory for slender beams and does not include warping of the cross-section shear deformation and or longitudinal inertia effects. It is given ~~in~~ by Love (76) and is discussed by Gere (32).

ii) If $K_t = P = 0$, Eq.(2.6) reduces to

$$GC_s \frac{\partial^2 \phi}{\partial z^2} - EC_w \frac{\partial^3 \phi}{\partial z^3} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.14}$$

This equation represents Timoshenko's torsion theory which includes the effect of warping of the cross section and has been treated in detail by Gere (32).

(iii) If $K_t = 0$, Eq.(2.6) reduces to

$$\left(GC_s - \frac{PI_p}{A}\right)\frac{\partial^2 \phi}{\partial z^2} - EC_w \frac{\partial^4 \phi}{\partial z^4} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2.15)$$

This equation represents the effect of an axial time-invariant compressive load added to Timoshenko's torsion theory.

(iv) If P = 0, Eq.(2.6) reduces to

$$GC_s \frac{\partial^2 \phi}{\partial z^2} - EC_w \frac{\partial^4 \phi}{\partial z^4} - K_t \phi - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2.16)$$

This equation represents the effect of Winkler type constant modulus elastic foundation added to Timoshenko Torsion theory.

2.6 NON-DIMENSIONALIZATION AND GENERAL SOLUTION OF EQUATION OF MOTION:

For mathematical simplification, it is convenient to to reduce Eq.(2.6) to a non-dimensional form, simultaneously introducing some dimensionless parameters having physical interpretations.

Introducing, $Z = z/L$, the non-dimensional beam length, and $\tau_1 = \left(\frac{EC_w}{\rho I_p L^4}\right)^{1/2} t$, the dimensionless time variable, Eq.(2.6) in non-dimensionless form can be written as:

$$\frac{\partial^4 \phi}{\partial Z^4} - (K^2 - \Lambda^2)\frac{\partial^2 \phi}{\partial Z^2} + 4\gamma^2 \phi + \frac{\partial^2 \phi}{\partial \tau_1^2} = 0 \qquad (2.17)$$

where

$$K^2 = \frac{GC_s L^2}{EC_w}$$, warping regidity parameter, (2.18)

$$\Delta^2 = \frac{PI_pL^2}{AEC_w}\text{, axial load parameter,} \tag{2.19}$$

and

$$\zeta^2 = \frac{K_tL^4}{4EC_w}\text{, foundation parameter,} \tag{2.20}$$

The general solution of Eq.(2.17) can be obtained by using the standard method of separation of variables. Thus, by taking ϕ in the form

$$\phi = X\,(Z)\,T\,(t_1) \tag{2.21}$$

and then substituting into Eq.(2.17), separating the variables, and setting the resulting expressions equal to $-\lambda_n^2$, we obtain

$$T = A_n \cos\lambda_n \bar{t}_1 + B_n \sin\ _n \bar{t}_1 \tag{2.22}$$

The expression for a normal mode of vibration is then

$$\phi = X\,(A_n \cos\lambda_n \bar{t}_1 + B_n \sin\lambda_n \bar{t}_1) \tag{2.23}$$

in which X is the normal function giving the shape of the mode of vibration and λ_n is the dimensionless torsional frequency parameter given by

$$\lambda_n^2 = \frac{\rho I_p L^4 p_n^2}{EC_w}, \tag{2.24}$$

Where p_n is the natural frequency of vibration in radious per unit of time. Any actual motion of the vibrating beam can be obtained by a summation of normal modes, so that in the general case

$$\phi = \sum_{n=1}^{\infty} X_n (A_n \cos \lambda_n \bar{t}_1 + B_n \sin \lambda_n \bar{t}_1) \qquad (2.25)$$

in which the coefficients A_n and B_n are found from the initial conditions of the vibration.

The equation for determining the normal function X, found by substituting Eq.(2.24) into the differential Eq.(2.17), is then

$$\frac{d^4X}{dZ^4} - (K^2 - \Delta^2)\frac{d^2X}{dZ^2} + (4\zeta^2 - \lambda_n^2)\,X = 0 \qquad (2.26)$$

The general solution of this equation may be found by taking the normal function X in the form:

$$X = D' e^{\bar{\eta} Z} \quad , \qquad (2.27)$$

which yields the auxiliary algebraic equation:

$$\bar{\eta}^4 - (K^2 - \Delta^2)\bar{\eta}^2 + (4\zeta^2 - \lambda_n^2) = 0 \qquad (2.28)$$

The four roots of the equation are

$$\bar{\eta}_1 = +\alpha_1, \ \bar{\eta}_2 = -\alpha_1; \ \bar{\eta}_3 = +i\beta_1, \ \bar{\eta}_4 = -i\beta_1 \qquad (2.29)$$

in which α_1 and β_1 are the positive, real quantities given by

$$\alpha_1 = (1/\sqrt{2})\left\{(K^2 - \Delta^2) + \left[(K^2 - \Delta^2)^2 + 4(\lambda_n^2 - 4\zeta^2)\right]^{1/2}\right\}^{1/2} \qquad (2.30)$$

and

$$\beta_1 = (1/\sqrt{2})\left\{-(K^2 - \Delta^2) + \left[(K^2 - \Delta^2)^2 + 4(\lambda_n^2 - 4\zeta^2)\right]^{1/2}\right\}^{1/2} \qquad (2.31)$$

The general solution of Eq.(2.26) then becomes either

$$X = D_1' e^{+\alpha_1 Z} + D_2' e^{-\alpha_1 Z} + D_3' e^{+i\beta_1 Z} + D_4' e^{-i\beta_1 Z}$$

or

$$X = D_1 \cosh \alpha_1 Z + D_2 \sin h\alpha_1 Z + D_3 \cos \beta_1 Z + D_4 \sin \beta_1 Z \quad (2.32)$$

There are four orbitrary constants in this expression which must be determined so as to satisfy the particular boundary conditions of the problem. For any beam there will be two boundary conditions at each end and these four conditions determine the frequency equation and the ratios of three of the constants to the fourth constant. Solving the frequency equation then determines the principal frequencies of vibration. With the frequencies and normal functions determined, the solution is essentially complete.

2.7 FREQUENCY EQUATIONS AND MODEL FUNCTIONS:

In this section, frequency equations and mode shapes for some special cases are are established. Gere's results (34) are obtained for the special case $\triangle^2 = \gamma^2 = 0$. Because of the complexity of the frequency equations, the discussion of the results is limited to the case of simply supported beam.

BOUNDARY CONDITIONS: In section (2.4a) natural boundary conditions were discussed. By combining these conditions in pairs, many types of single-span beams can be analyzed. In terms of non-dimensional parameters, the boundary conditions can be written as:

1. Simple Support:

$$X = 0, \quad \frac{d^2X}{dZ^2} = 0 \qquad (2.33)$$

2. Fixed Support:

$$X = 0, \quad \frac{dX}{dZ} = 0 \qquad (2.34)$$

3. Free End:

$$\frac{d^2X}{dZ^2} = 0, \quad (K^2 - \triangle^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \qquad (2.35)$$

Before we proceed to derive the frequency and Normal mode equations for various cases, from Equations (2.30) and (2.31) we obtain:

$$\alpha_1^2 = (K^2 - \triangle^2) + \beta_1^2 \qquad (2.36)$$

and

$$\lambda_n^2 = \alpha_1^2\ \beta_1^2 + 4\zeta^2 \qquad (2.37)$$

If in case, the beam is not vibrating and only elastic torsional buckling is to be investigated the expressions for α_1 and β_1 from Equations (2.30) and (2.31) reduce to:

$$\alpha_1 = (1/\sqrt{2})\left\{(K^2 - \triangle^2) + \left[(K^2 - \triangle^2)^2 - 16\zeta^2\right]^{1/2}\right\}^{1/2} \qquad (2.38)$$

and

$$\beta_1 = (1/\sqrt{2})\left\{-(K^2 - \triangle^2) + \left|(K^2 - \triangle^2)^2 - 16\zeta^2\right|^{1/2}\right\}^{1/2} \qquad (2.39)$$

The following frequency equations which we derive for various cases are also useful in finding the torsional buckling loads when the reduced Equations (2.38) and (2.39) are used for α_1 and β_1 respectively. In this case the following relations are to be used:

$$\alpha_1^2 = -4\zeta^2/\beta_1^2 \qquad (2.40)$$

and

$$\triangle^2 = K^2 + \beta_1^2 - \alpha_1^2 \qquad (2.41)$$

2.7.1 SIMPLY SUPPORTED BEAM:

This is the simplest case which admits complete analytical treatment. An example is a beam supported by framing angle connections at the two ends. These beams are used in building construction and therefore are of practical importance.

The boundary conditions from Equations (2.33) are:

$$X = d^2X/dZ^2 = 0 \quad \text{at } Z = 0$$

and

$$X = d^2X/dZ^2 = 0 \quad \text{at } Z = 1$$

For the conditions at $Z = 0$, Equation (2.32) gives:

$$D_3 + D_1 = 0,$$

and $D_1(\alpha_1^2 + \beta_1^2) = 0.$

Since the secular determinant $\alpha_1^2 + \beta_1^2 \neq 0$, it follows that

$$D_1 = D_3 = 0 \,. \tag{2.42}$$

From the second pair of conditions, Equation (2.32) gives:

$$D_2 \sinh \alpha_1 + D_4 \sin \beta_1 = 0, \tag{2.43}$$

and

$$D_2\, \alpha^2 \sinh \alpha_1 - D_4\, \beta^2 \sin \beta_1 = 0 \tag{2.44}$$

For a non-trivial solution, the secular determinant must vanish. This gives the characterestic equation

$$(\alpha_1^2 + \beta_1^2) \sinh \alpha_1 \sin \beta_1 = 0$$

Since $\alpha_1^2 + \beta_1^2 \neq 0$, and $\sinh \alpha_1 \neq 0$, we obtain the frequency equation for this case as:

$$\sin \beta_1 = 0 \tag{2.45}$$

From Equation (2.45) we have,

$$\beta_1 = n\pi,\ n = 1,2,3,\ldots\ldots \tag{2.46}$$

This is the frequency equation for a simply supported beam and by using the relations (2.36) and (2.37), we find the expression for the frequency parameter λ_n as:

$$\lambda_n = \left| n^2\pi^2(n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2 \right|^{1/2} \tag{2.47}$$

Since $\sin \beta_1 = 0$, we find from Equation (2.43) or (2.44) that $D_2 = 0$. Hence the model function is

$$X = D_4 \sin n\pi Z \tag{2.48}$$

The complete expression for the angle of twist ϕ is obtained by summing up the normal modes, so that

$$\phi = \sum_{n=1}^{\infty} \sin n\pi Z (A_n \cos \lambda_n t_1 + B_n \sin \lambda_n t_1)$$

in which A_n and B_n are determined by the initial conditions.

Gere (31) studied the influence of warping parameter K, and concluded that it increased the frequency of vibration as warping increases the stiffness of the bar against rotation. For small values of K, which means C_w is relatively large, the effect of warping is considerable and must be taken into account. For large K, which means C_w is relatively small, the warping effect is also small and may be neglected in many cases.

To estimate the individual influences of axial load and elastic foundation, Equation (2.47) can be reduced in the following manner.

(a) If the effect of axial load alone is to be studied, by putting $\gamma = 0$, we obtain

$$\lambda_1 = n\pi(n^2\pi^2 + K^2 - \Delta^2)^{1/2} \tag{2.49}$$

(b) If the influence of elastic foundation alone is to be investigated, by putting $\Delta = 0$, we get

$$\lambda_2 = \left[n^2\pi^2 \, (n^2\pi^2 + K^2) + 4\gamma^2 \right]^{1/2} \tag{2.50}$$

(c) If the both the effects of axial load and elastic foundation are to be neglected, by putting $\Delta = 0$ and $\gamma = 0$, we obtain the equation that was derived by Gere (31) as:

$$\lambda_3 = n\pi \, (n^2\pi^2 + K^2)^{1/2} \tag{2.51}$$

Denoting by r_1 the ratio of the frequency of vibration with axial load alone considered, Equation (2.49), to the frequency with axial load also neglected, Equation (2.51), we obtain

$$r_1 = \frac{\lambda_1}{\lambda_3} = \left[1 - \frac{\Delta^2}{n^2\pi^2 + K^2} \right]^{1/2} \tag{2.52}$$

Similarly, denoting by r_2 the ratio of the frequency of vibration with elastic foundation alone considered, Equation (2.50), to the frequency with elastic foundation also neglected, Equation (2.51), we obtain

$$r_2 = \frac{\lambda_2}{\lambda_3} = \left| 1 + \frac{4\gamma^2}{n^2\pi^2(n^2\pi^2 + K^2)} \right|^{1/2} \qquad (2.53)$$

To find the combined influence of axial load and elastic foundation, let us denote by r_3 the ratio of the frequency of vibration with both axial load and elastic foundation considered, Equation (2.47), to the frequency with both axial load and elastic foundation neglected, we obtain

$$r_3 = \frac{\lambda_n}{\lambda_3} = \left| 1 + \frac{4\gamma^2 - n^2\pi^2\Delta^2}{n^2\pi^2(n^2\pi^2 + K^2)} \right|^{1/2} \qquad (2.54)$$

Fig.2.2 shows the variation of r_1 with Δ, for values of K = 0.1, 1.0 and 10.0 for the first fundamental mode of vibration. The effect of axial load is to decrease the frequency of vibration, since the axial load decreases the stiffness of the bar against rotation. For small Δ, which means axial load P is relatively small, the effect of axial load is small and for large Δ, which means P is relatively large, the effect of axial load is quite considerable.

Figs.2.3 and 2.4 show the variation of r_2 with γ, for values of K = 1 and 10 respectively, for the first three modes of vibration. The effect of the elastic foundation is to increase the

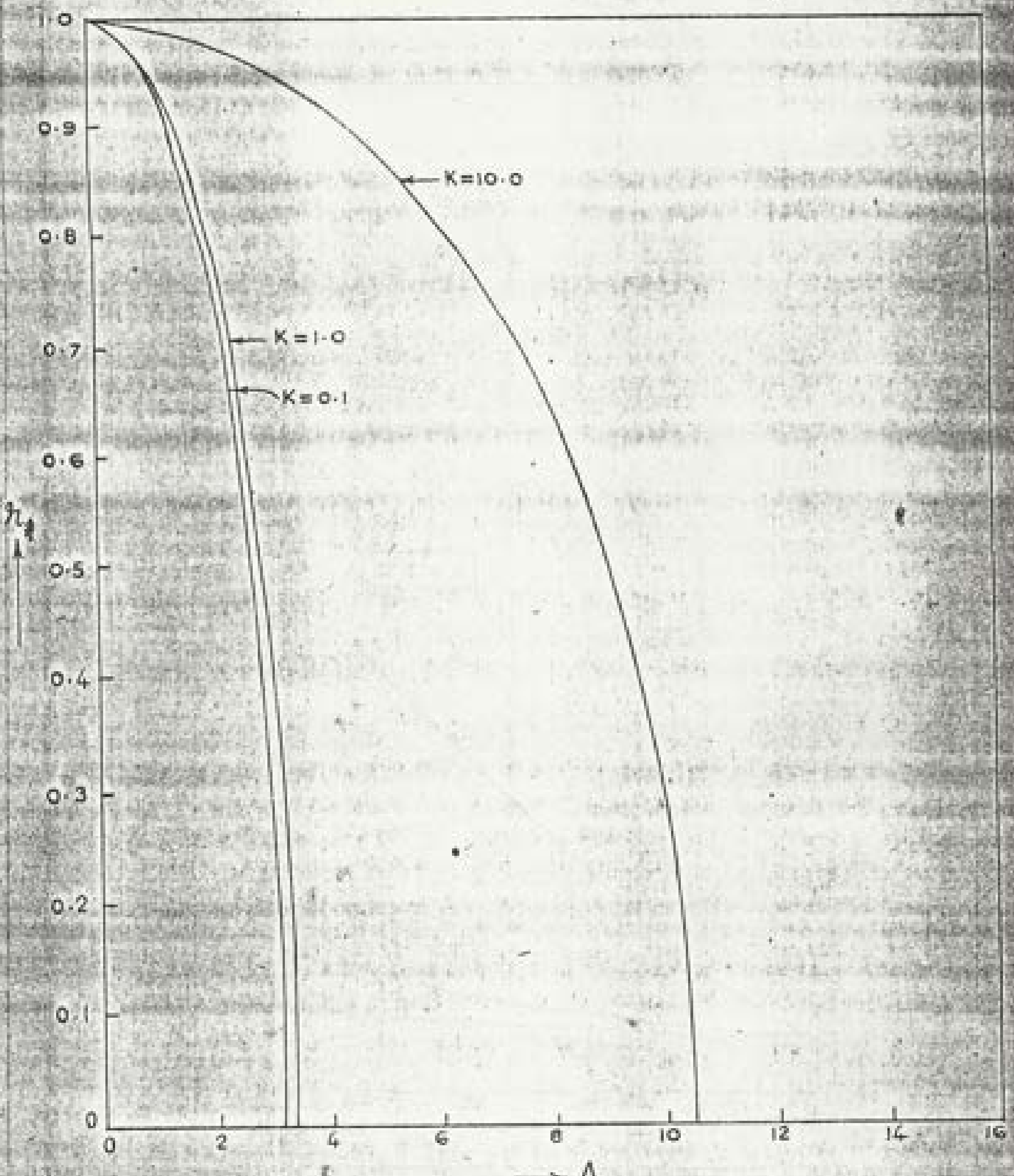


Fig. 2.2 Variation of η_1 with Δ, for Values of K=0.1, 1.0 and 10.0 (n=1)

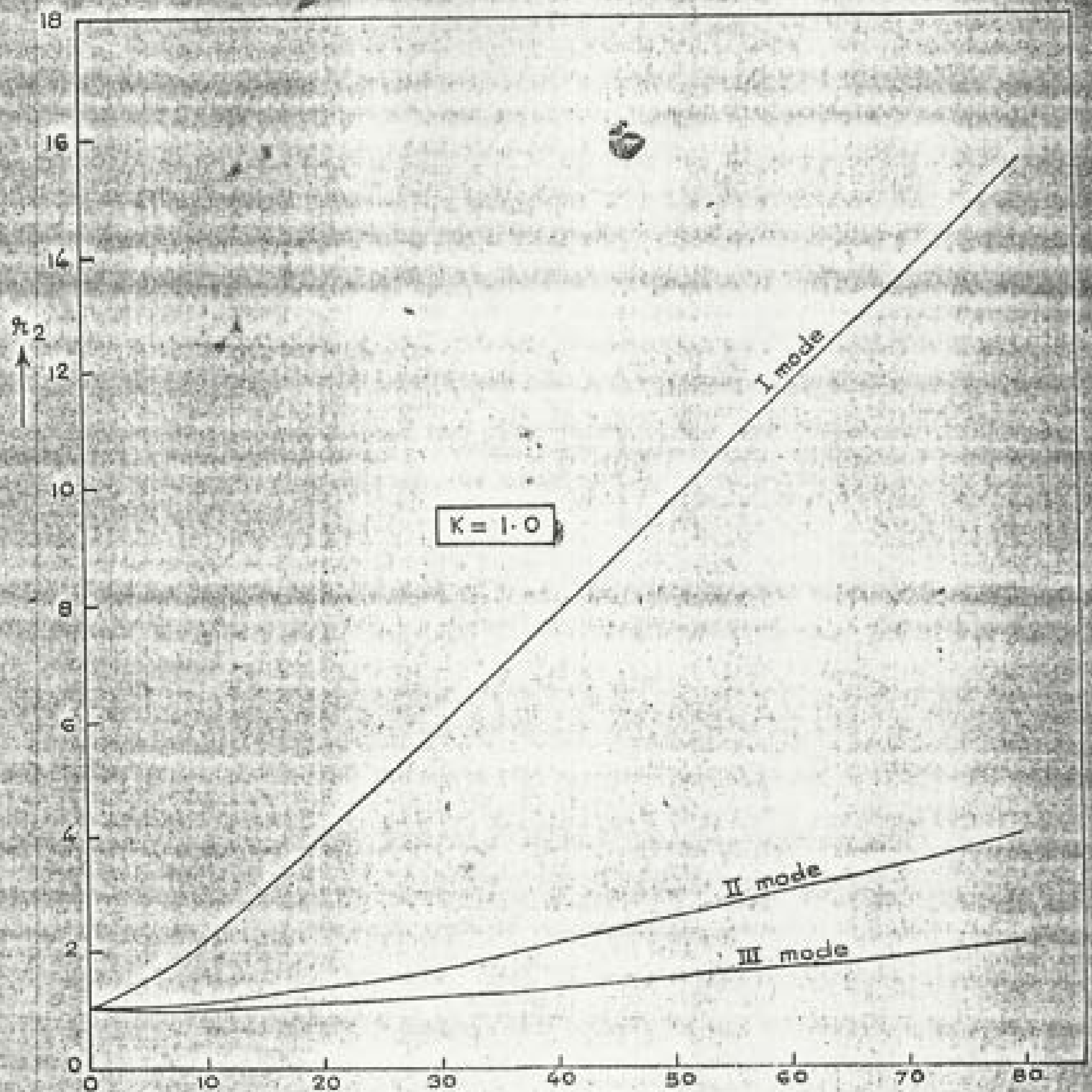


Fig. 2·3. Variation of κ_2 with γ for K=1·0 for the first three modes of Vibration

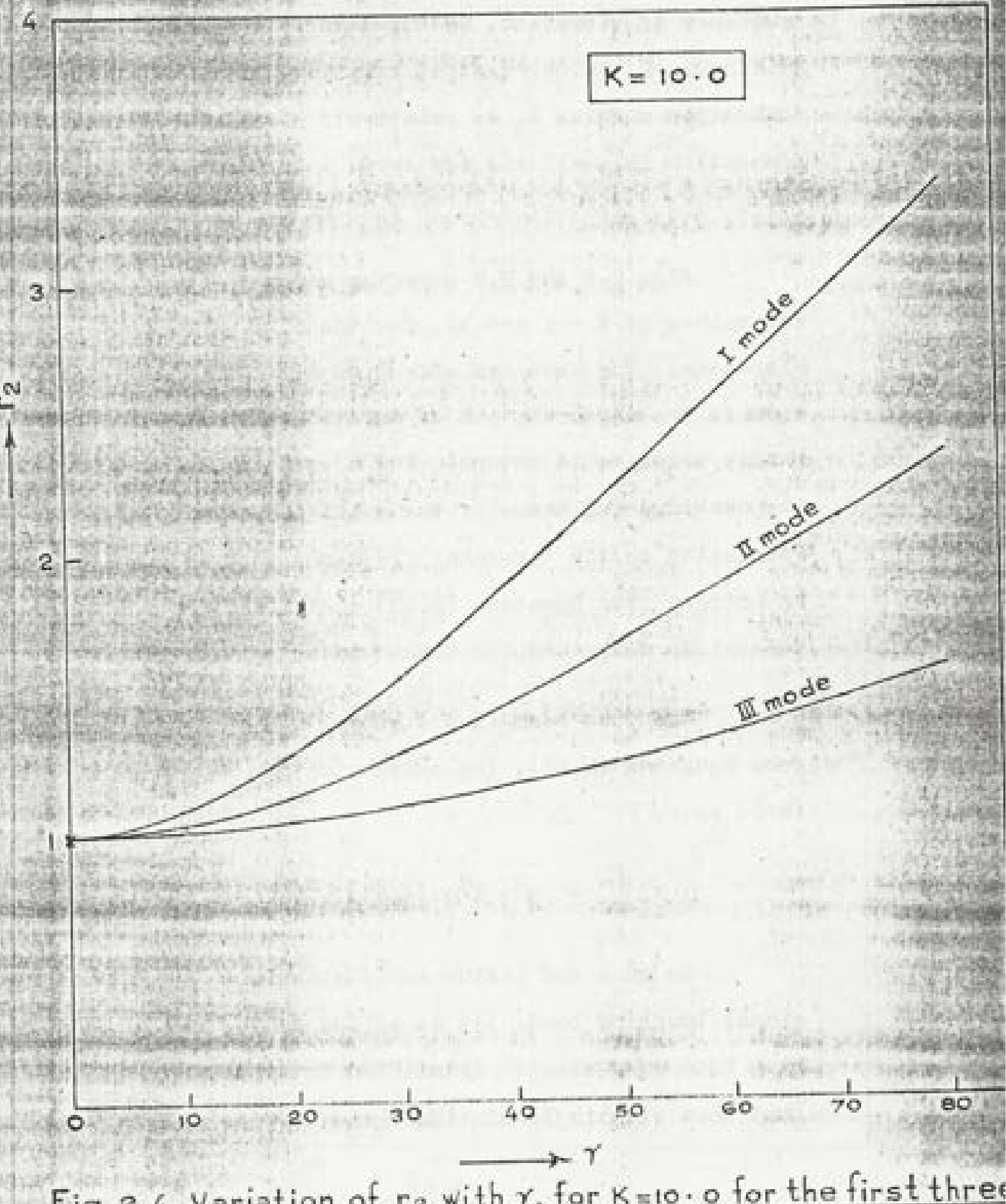


Fig. 2.4. Variation of r_2 with γ, for K=10·0 for the first thre
modes of vibration.

frequency of vibration, as the elastic foundation increases the stiffness of the bar against rotation. For small γ, which means foundation modulus K_t is relatively small, the effect of elastic foundation is small and for large γ, which means K_t is relatively large, the effect of elastic foundation is quite considerable.

Figs.2.5 and 2.6 show the variation of r_3 with Δ and γ, for values of K = 1 and 10, for the first fundamental mode of vibration. The combined effect of axial load and elastic foundation is the algebraic sum of individual influences which are actually opposite in nature. For a value of $\gamma^2 = 0.25\, n^2\pi^2\Delta^2$, the combined influence of the axial compressive load and elastic foundation on the torsional frequency becomes zero. It can also be noticed from Equation (2.53) that the influence of elastic foundation decreases for higher modes of vibration.

When the beam is not vibrating, ie., $\lambda = 0$, we obtain from Equation (2.47), the expression for torsional buckling load (n=1) as,

$$\Delta^2_{cr} = \pi^2 + K^2 + (4/\pi^2)\gamma^2 \tag{2.55}$$

To show the influence of elastic foundation on the torsional buckling load, let us define by r_4, the ratio of the buckling load when elastic foundation is considered, to the buckling load when elastic foundation is neglected.

$$r_4 = 1 + \frac{4\gamma^2}{\pi^2(\pi^2 + K^2)} \tag{2.56}$$

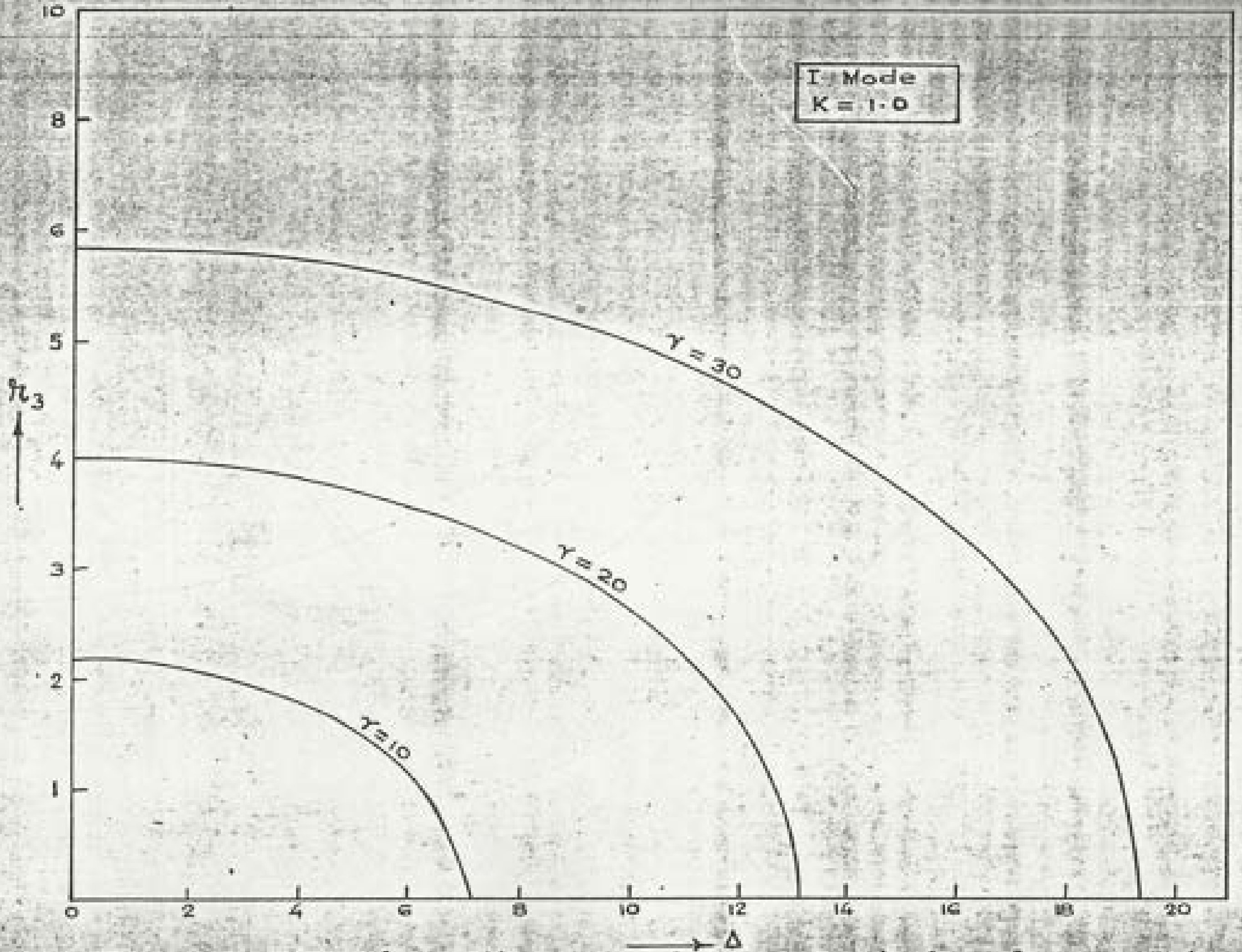


Fig. 2.5. Variation of π_3 with Δ and γ, for K = 1·0 for the first fundamental mode of Vibration.

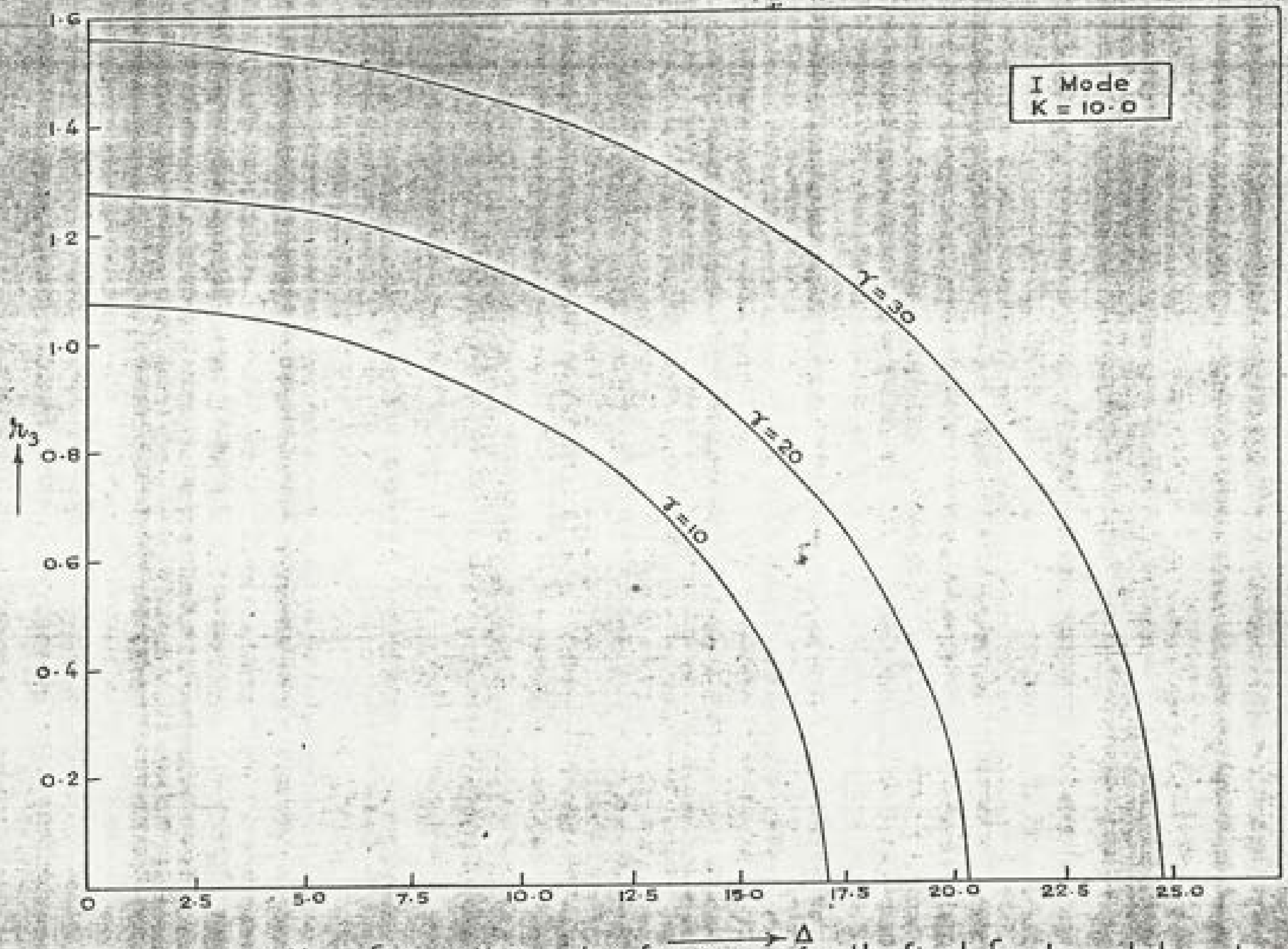


Fig. 2.6. Variation of λ_3 with Δ and γ, for K = 10·0 for the first fundamental mode of Vibration.

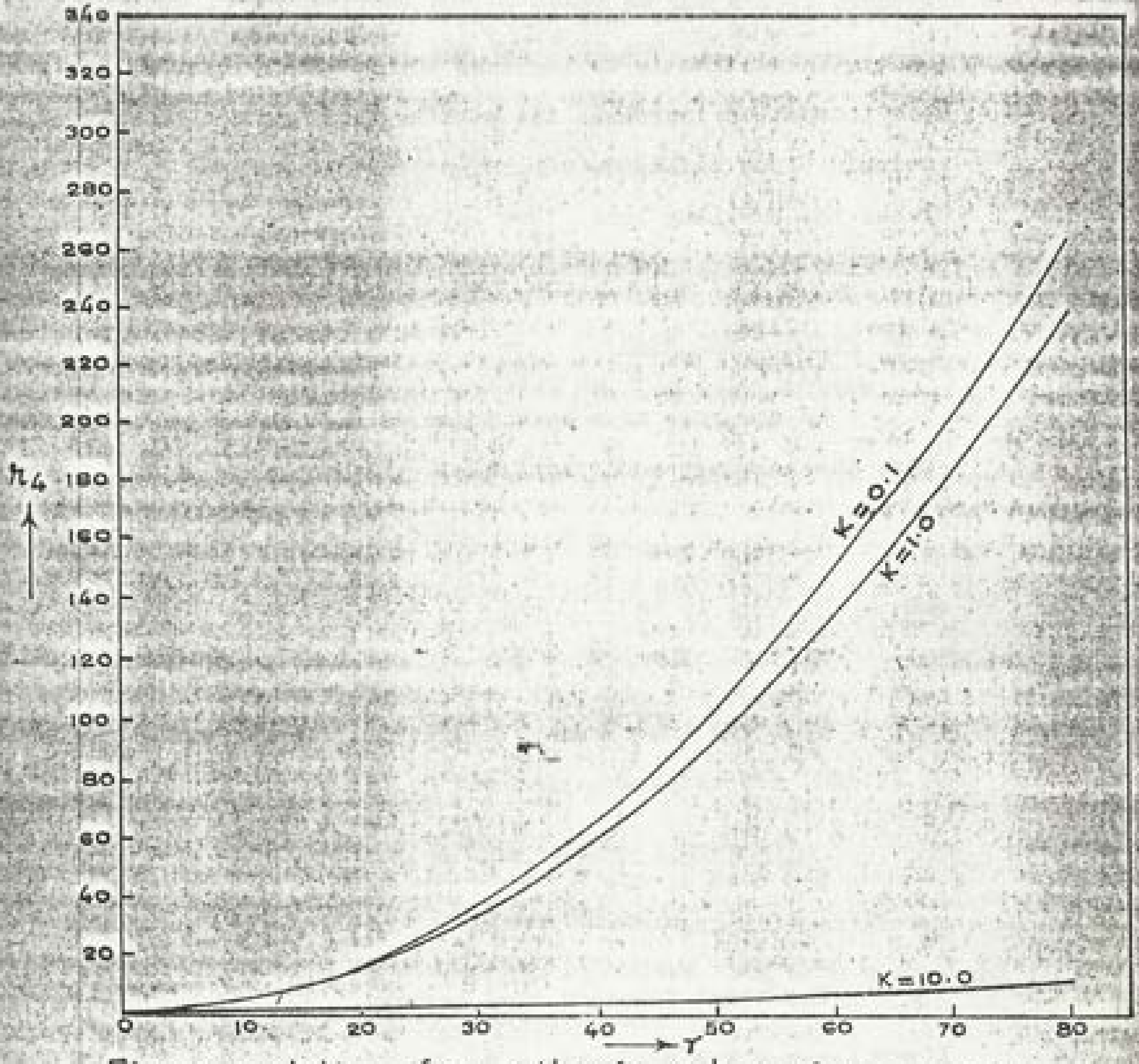


Fig. 2·7. Variation of π_4 with γ for Values of K=0·1, 1·0 and 10·0.

From the above Eq.(2.56) and Fig.2.7, which shows the variation of r_4 with γ for values of K = 0.1, 1.0 and 10.0, it can be observed that in the case of torsional buckling also the effect of elastic foundation is to increase the buckling load, as the elastic foundation increases the stiffness of the member against rotation. The influence of the warping parameter K is also to increase the buckling load. But relatively, the effect of warping parameter is more pronounced than that of elastic foundation.

2.72 FIXED-FIXED BEAM:

In the case of a beam which is built-in rigidly at both ends, the boundary conditions are:

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 1$$

Applying the boundary conditions to the general solutions, Eq.(2.32), frequency equation can be obtained as,

$$2-2\cosh\alpha_1\cos\beta_1 + \frac{(\alpha_1^2-\beta_1^2)}{\alpha_1\beta_1}\sinh\alpha_1\sin\beta_1 = 0 \qquad (2.57)$$

The modal function then becomes,

$$X = D_1(\cosh\alpha_1 Z + \beta_1\eta_1\sinh\alpha_1 Z - \cos\beta_1 Z - \alpha_1\eta_1\sin\beta_1 Z) \qquad (2.58)$$

where

$$\eta_1 = \frac{\cos\beta_1 - \cosh\alpha_1}{\beta_1\sinh\alpha_1 - \alpha_1\sin\beta_1} = \frac{\beta_1\sin\beta_1 + \alpha_1\sinh\alpha_1}{\alpha_1\beta_1(\cos\beta_1 - \cosh\alpha_1)} \qquad (2.59)$$

2.7.3. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

With the end Z = 0, taken as the simply supported end, and the end Z = 1 as the built-in end, the boundary conditions are:

$$X = \frac{d^2X}{dZ^2} = 0 \quad \text{at } Z = 0,$$

and

$$X = \frac{dX}{dZ} = 0 \quad \text{at } Z = 1.$$

The frequency equation in this case becomes

$$\beta_1 \tanh \alpha_1 - \alpha_1 \tan \beta_1 = 0 \tag{2.60}$$

The modal function then is

$$X = D_2(\sinh \alpha_1 Z - \eta_2 \sin \beta_1 Z) \tag{2.61}$$

where

$$\eta_2 = \frac{\sinh \alpha_1}{\sin \beta_1} = \frac{\alpha_1 \cosh \alpha_1}{\beta_1 \cos \beta_1} \tag{2.62}$$

2.7.4. CANTILEVER BEAM WITH WARPING RESTRAINED:

For a cantilever beam built-in rigidly at the end Z=0 so that warping is completely prevented, and with a free end Z at Z = 1, the boundary conditions are:

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 1$$

The frequency equation for this beam can be obtained as:

$$2 + \frac{\alpha_1^4 + \beta_1^4}{\alpha_1^2\ \beta_1^2} \cosh \alpha_1 \ \cos \beta_1 + \frac{\alpha_1^2 - \beta_1^2}{\alpha_1\ \beta_1} \sinh \alpha_1 \ \sin \beta_1 = 0 \qquad (2.63)$$

The modal function then becomes,

$$X = D_1(\cosh \alpha_1 Z + \beta_1 \eta_3 \sinh \alpha_1 Z - \cos \beta_1 Z - \alpha_1 \eta_3 \sin \beta_1 Z) \qquad (2.64)$$

where

$$\eta_3 = \frac{\alpha_1 \ \sin \beta_1 - \beta_1 \ \sinh \alpha_1}{\alpha_1^2 \ \cos \beta_1 + \beta_1^2 \ \cos \alpha_1} =$$

$$= - \frac{\beta_1^2 \ \cos \ \beta_1 + \alpha_1^2 \ \cosh \alpha_1}{\alpha_1 \beta_1 (\beta_1 \sin \ \beta_1 + \alpha_1 \sinh \alpha_1} \qquad (2.65)$$

2.7.5. CANTILEVER BEAM WITH UNRESTRAINED WARPING:

In the previous case, a cantilever beam was considered in which the supported end was fixed and offered complete restraint against warping. A cantilever beam may also be supported in a manner such that warping is free to occur at the supported end. An example is a cantilever beam supported by the ordinary framing angles and moment resistant connections used in building construction. With regard to torsion, such a support offers restraint against rotation but not warping and hence is a simple support. It is, of course, a fixed support with regard to bending.

Thus, for a cantilever simply supported at one end and free at the other, the boundary conditions are:

$$X = \frac{d^2X}{dZ^2} = 0 \quad \text{at } Z = 0$$

and

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 1$$

Applying the above boundary conditions, the frequency equation can be obtained as,

$$\alpha_1^3 \tanh \alpha_1 - \beta_1^3 \tan \beta_1 = 0 \tag{2.66}$$

The modal function in this case becomes,

$$X = D_2(\sinh \alpha_1 Z + \eta_4 \sin \beta_1 Z) \tag{2.67}$$

where

$$\eta_4 = \frac{\alpha_1^2 \sinh \alpha_1}{\beta_1^2 \sin \beta_1} = \frac{\beta_1 \cosh \alpha_1}{\alpha_1 \cos \beta_1} \tag{2.68}$$

2.7.6. BEAM WITH FREE ENDS:

In the case of a beam which is free at both ends, the boundary conditions are:

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 0$$

and

$$\frac{d^2X}{dZ^2} = (K^2 - \Delta^2)\frac{dX}{dZ} - \frac{d^3X}{dZ^3} = 0 \quad \text{at } Z = 1$$

The frequency equation for this case becomes,

$$2 - 2\cosh \alpha_1 \cos \beta_1 + \frac{\beta_1^6 - \alpha_1^6}{\alpha_1^3\ \beta_1^3} \sinh \alpha_1 \sin \beta_1 = 0 \tag{2.69}$$

The modal function therefore becomes

$$X = D_1(\cosh \alpha_1 Z + \eta_5 \sinh \alpha_1 Z + (\alpha_1/\beta_1)^2 \cos \beta_1 Z + (\beta_1/\alpha_1)\eta_5 \sin \beta_1 Z) \qquad (2.70)$$

where

$$\eta_5 = \frac{\alpha_1^3(\cos \beta_1 - \cosh \alpha_1)}{\alpha_1^3 \sinh \alpha_1 - \beta_1^3 \sin \beta)} = \frac{\beta_1^3 \sinh \alpha_1 + \alpha_1^3 \sin \beta_1}{\beta_1^3(\cos \beta_1 - \cosh \alpha_1)} \qquad (2.71)$$

2.8. RESULTS AND DISCUSSION:

The frequency equations derived in this section for various combinations of boundary conditions are highly transcendental in nature and can be solved only by lengthy trial-and-error procedure. As is stated earlier the same frequency equations can be used to obtain the Elastic Torsional Buckling loads for various end condition but with the only difference that for α_1 and β_1, Equations (2.38) and (2.39) are to be used in conjunction with Equations (2.40), (2.41) and the corresponding frequency Equation. A computer program has been written in Fortran IV for solution of the above Frequency equations on IBM-1130 computer at the Computer Center, Andhra University, Waltair. Typical results for simply supported, fixed-fixed beam and beam fixed at one end and simply supported at the other for the fundamental mode (n=1) for values of K=1 and 10 are presented in Figs. 2.8 to 2.12 showing the combined influence of axial load (Δ) and Elastic foundation (γ). The individual influences also can be easily observed from these graphs. Figs 2.8 and 2.9 show the variation of the fundamental torsional frequency parameter λ_n^2(n=1), for a simply supported beam,

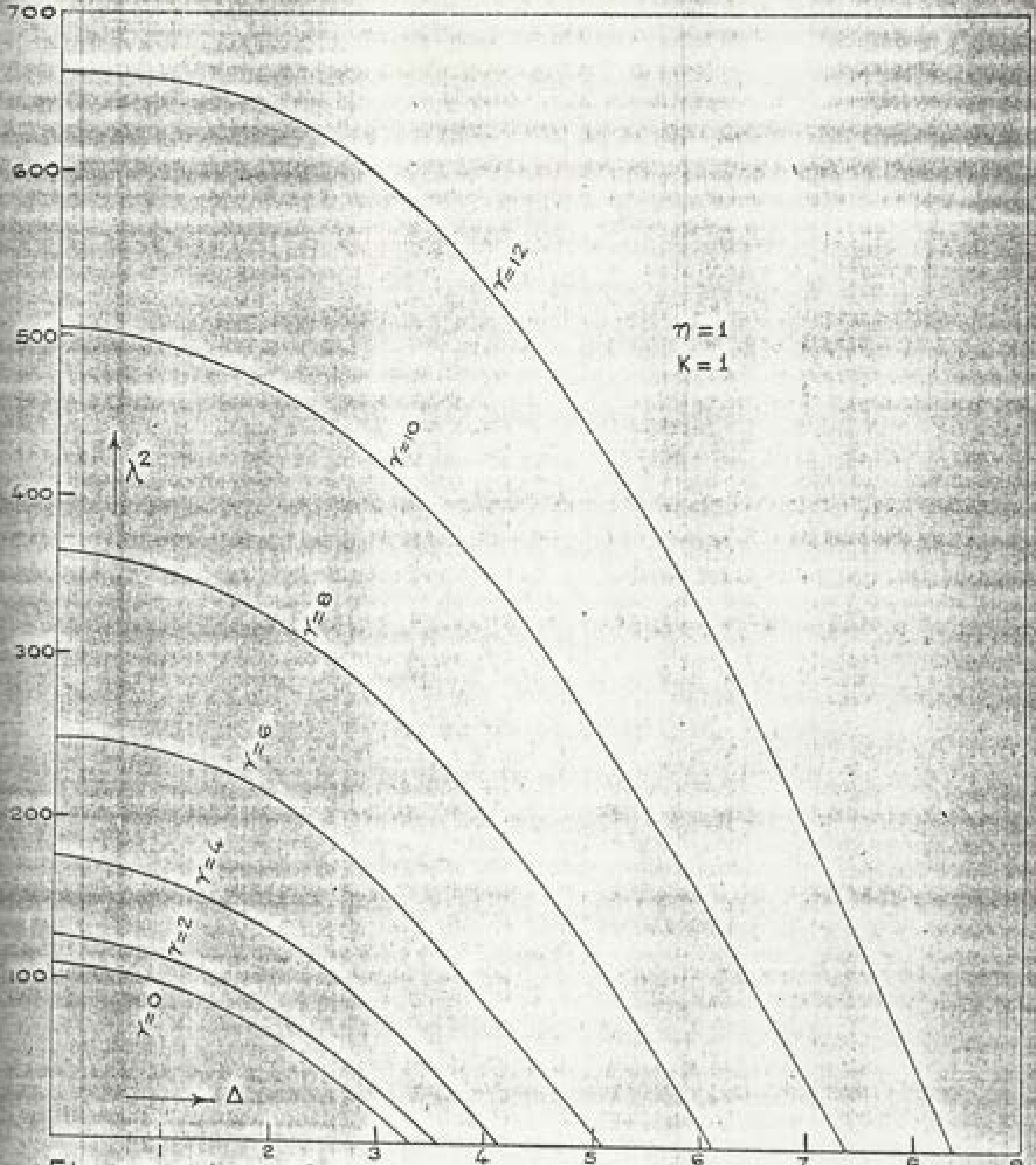
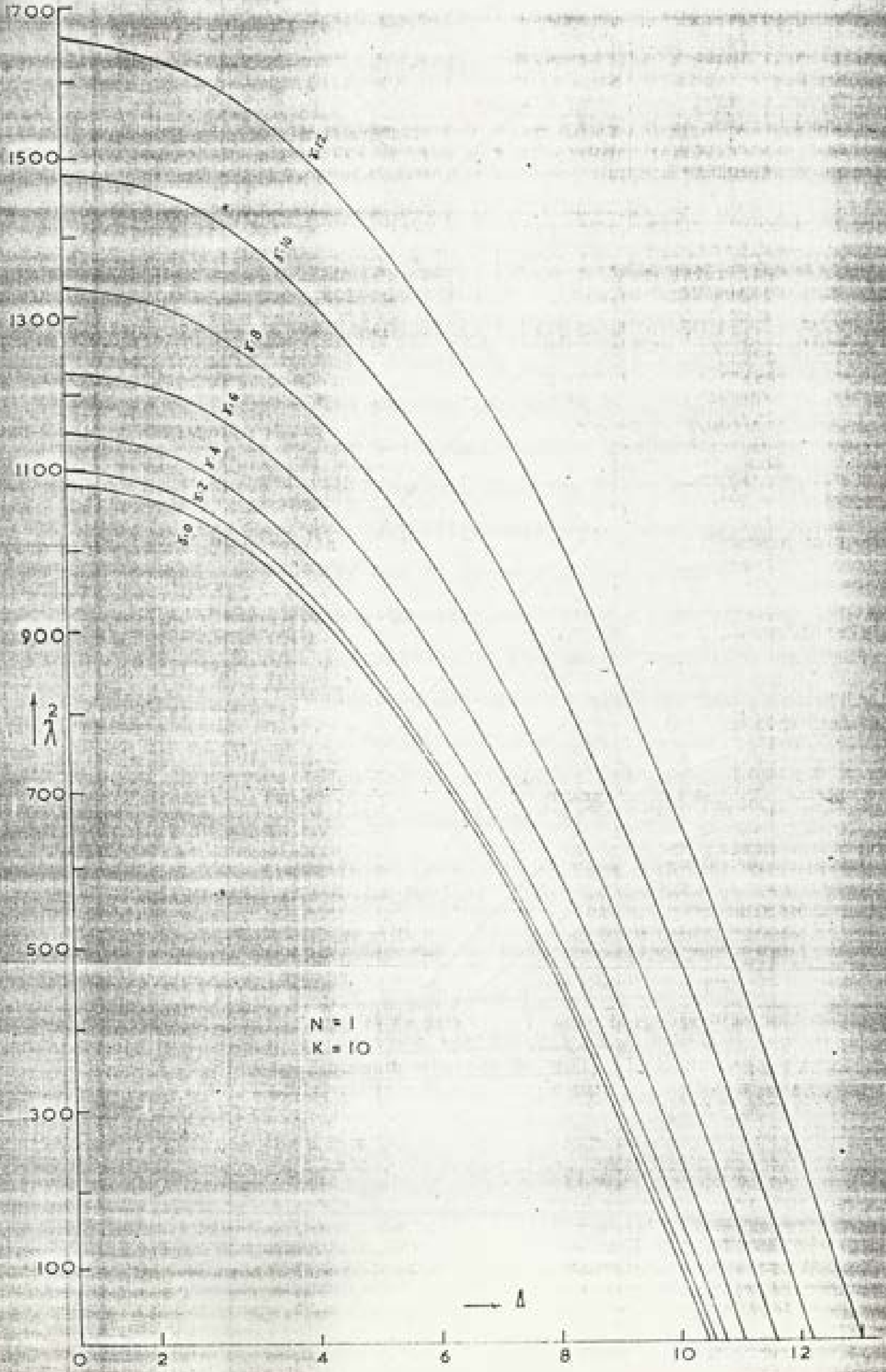


Fig. 2·8 Values of frequency & critical buckling load parameters for a simple supported beam.



9

FIG.2.9. VALUES OF FREQUENCY & CRITICAL BUCKLING PARAMETER Δ FOR A SIMPLY SUPPORTING BEAM.

with various values of load parameter Δ and foundation parameter γ for values of K = 1 and 10 respectively. Figs.2.10 and 2.11 show the results for fixed-fixed beam and, the results corresponding to a beam fixed at one end and simply supported at the other are shown in Figs.2.12 and 2.13.

It can be observed from these graphs that the values of the critical buckling loads for various values of γ can be obtained from the graphs for $\lambda^2 = 0$ ie., from the axis on which is taken. ~~When the~~ For no axial load ~~is not existing~~ the values of the frequency parameter λ^2 can be obtained from these graphs for $\Delta = 0$ ie., from the vertical axis on which λ^2 is plotted for various values of γ. The combined influence of the foundation parameter γ and the load parameter Δ can be observed from the graphs to be due to the interaction between the individual influences on the frequency of vibration, which are interestingly opposite in nature. Independently as the load parameter increases the frequency parameter decreases to zero. In the absence of axial load, the frequency increases for increasing values of γ. It can be therefore concluded that the combined influence of foundation and load parameters is the algebraic sum of the individual influences on the frequency of vibration.

2.9. APPROXIMATE SOLUTIONS BY GALERKIN'S TECHNIQUE:

Except for the simply-supported beam, the frequency equations for other boundary conditions derived in the above sections (2.7) and (2.8) can be observed to be highly transcendental

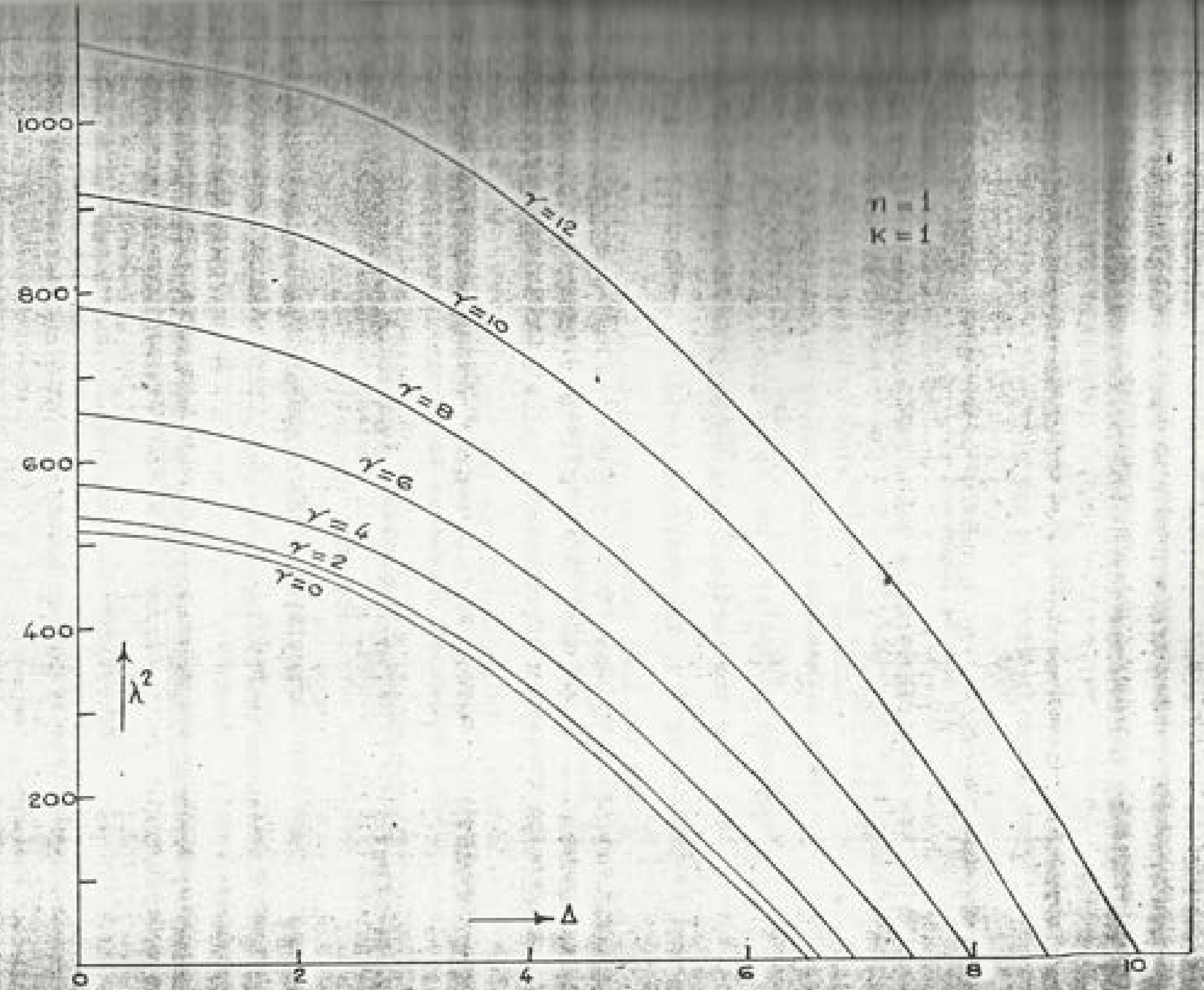


Fig 2.10. Values of frequency & critical buckling load parameters

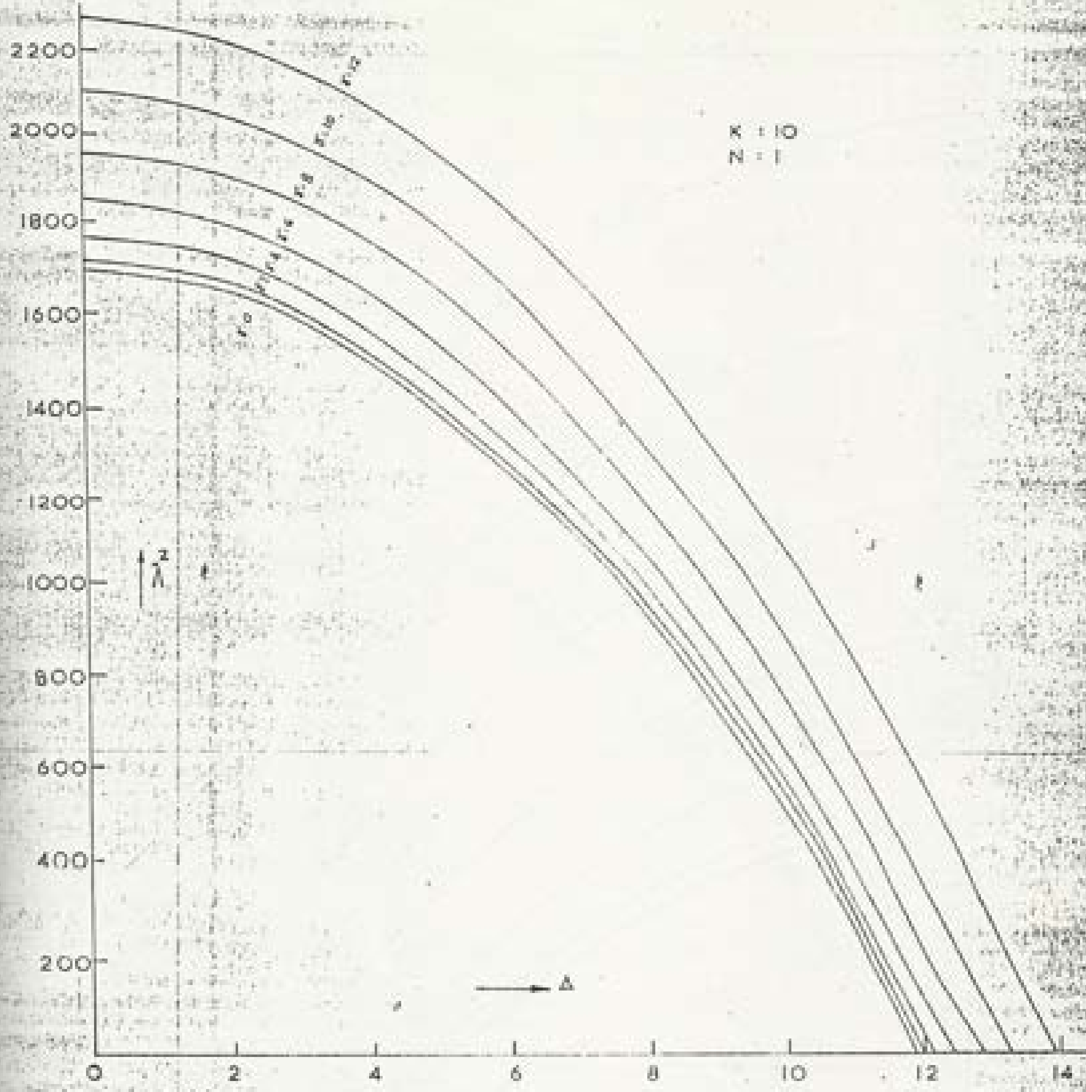


FIG.2-11.VALUES OF FREQUENCY & CRITICAL BUCKLING PARAMETERS FOR A FIXED – FIXED BEAM.

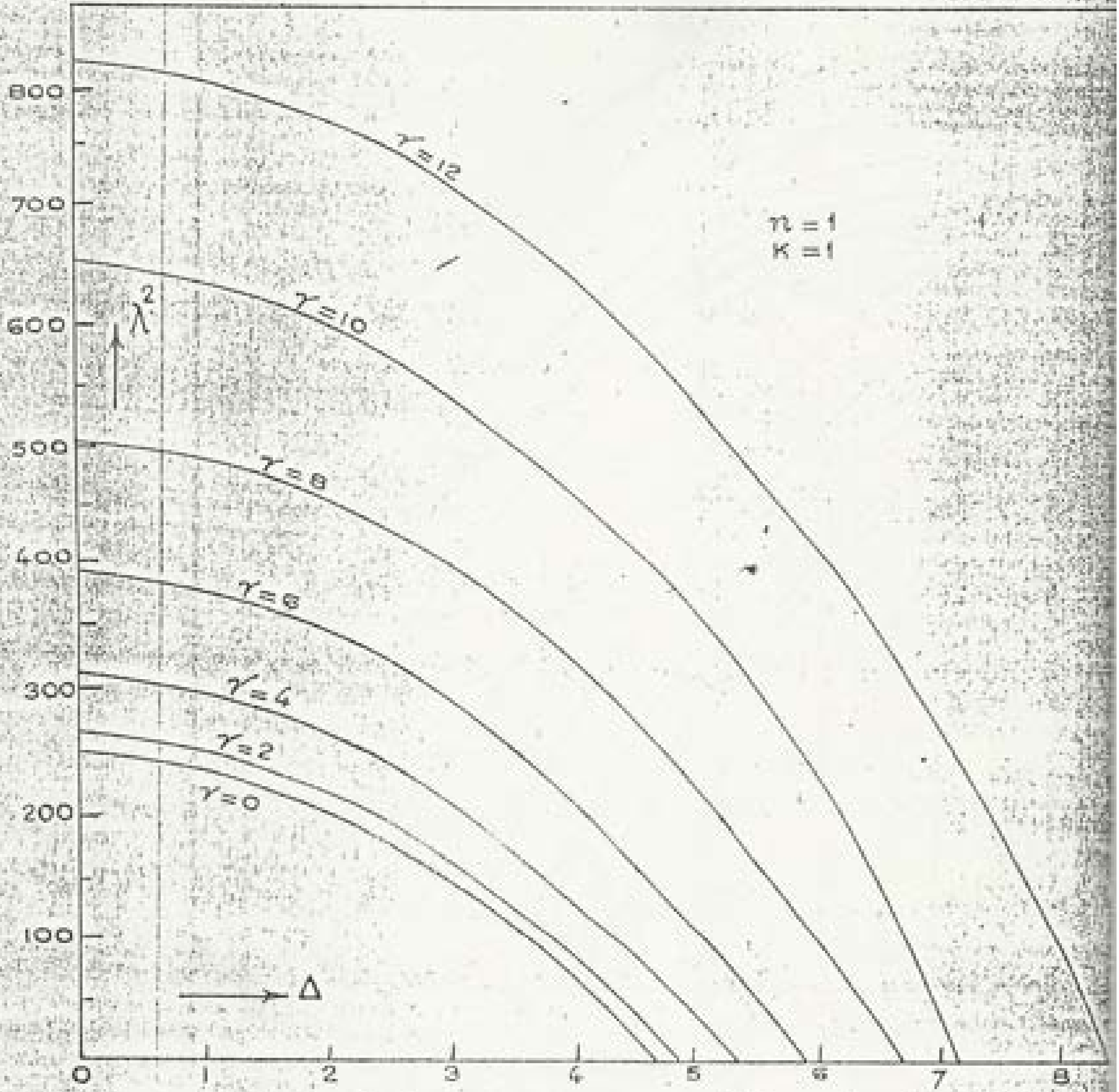


Fig. Values of frequency & critical buckling loads for a fi
simply supported beam.

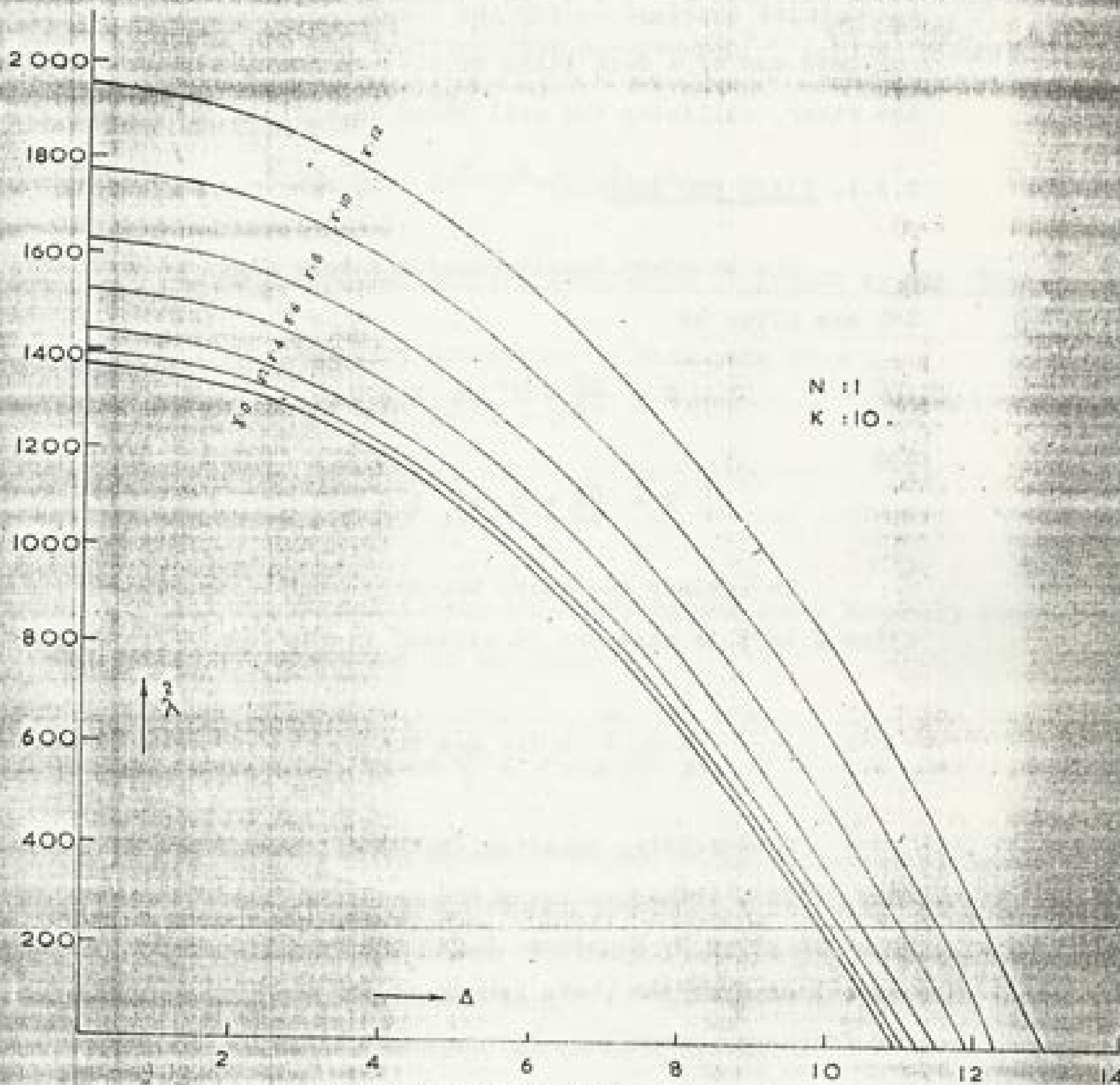


FIG.2.13 VALUES OF FREQUENCY & CRITICAL BUCKLING PARAMETERS FOR A FIXED-SIMPLY SUPPORTING BEAM.

and are solved on a digital computer only by lengthy-trial and error method. An attempt has been made in this section to derive approximate expressions for the torsional frequencies of fixed end beam and of a beam fixed at one end and simply supported at the other, utilizing the well known Galerkin's technique(77).

2.9.1. FIXED END BEAM:

The boundary conditions for a beam fixed at both ends, Z=0 and Z=1 are given by

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 1$$

To satisfy the above boundary conditions, the normal function X in this case can be assumed in the form

$$X = \sum_{n=1}^{\infty} B_n (1 - \cos 2n\pi Z) \tag{2.72}$$

Substituting Equation (2.72) in the differential equation (2.26), orthagonalizing the resulting error with the assumed function given by Equation (2.72) and integrating the obtained expression over the whole length of the beam, the expression for the frequency parameter λ, can be obtained as,

$$\lambda = 2 \left| (n^2\pi^2/3)(4n^2\pi^2 + K^2 - \gamma^2) + \Delta^2 \right|^{1/2} \tag{2.73}$$

In arriving Equation (2.73), only one term of the infinite series of Equation (2.72) is utilized. Hence, Equation(2.73) gives an upper bound for the natural frequency parameter .

By putting $\lambda = 0$, and $n = 1$, in Equation (2.73) the expression for the buckling load parameter Δ_{cr}^2, for the fixed end beam can be obtained as

$$\Delta_{cr}^{2} = 4\pi^2 + K^2 + (3/\pi^2)\gamma^2 \tag{2.74}$$

2.9.2. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

The boundary conditions in this case are:

$$X = \frac{dX}{dZ} = 0 \quad \text{at} \quad Z = 0$$

and

$$X = \frac{d^2X}{dZ^2} = 0 \quad \text{at} \quad Z = 1$$

The normal function satisfying the above boundary conditions can be assumed in the form

$$X = \sum_{n=1}^{\infty} C_n \left(\cos\frac{n\pi}{2} Z - \cos\frac{3n\pi}{2} Z\right) \tag{2.75}$$

Substituting Equation (2.75) in the differential Equation (2.26),orthagonalizing the resulting error with the assumed function given by Equation (2.75) and integrating the obtained expression over the whole length of the beam, the equation for the frequency parameter λ can be obtained as,

$$\lambda = \left[1{:}25\, n^2\pi^2 (2.05\, n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2\right]^{1/2} \tag{2.76}$$

Equation (2.76) also gives an upper bound for the natural torsional frequency parameter as only one term of the infinite series of

Equation (2.75) is utilized in obtaining the solution.

By putting $\lambda = 0$ and $n = 1$, in Equation (2.76), the expression for the buckling load parameter Δ_{cr}^{2}, for the beam fixed at one end and simply supported at the other can be obtained as

$$\Delta_{cr}^{2} = 2.05\,\pi^2 + K^2 + (3.2/\pi^2)\,\xi^2 \qquad (2.77)$$

Tables 2.1 and 2.2 show the comparison between the exact results (obtained by digital computer) and the approximate results (obtained by Galerkin's technique) of the frequency parameter λ for the first mode of vibration (n=1) of, fixed end beam and a beam fixed at one end and simply supported at the other respectively. The agreement between the results is quite good.

2.9.3. LIMITTING CONDITIONS:

The limit~~t~~ing conditions at which the combined influence of the axial compressive load and elastic foundation on the torsional frequency becomes zero, for some cases are as follows:

1) Simply-Supported Beam: From Equation (2.47) the limitting condition in this case becomes,

$$\gamma = 0.5\, n\pi\, \Delta \qquad (2.78)$$

2) Fixed-End Beam: From Equation (2.73) the limit~~t~~ing condition in this case is

$$\gamma = 0.574\, n\pi\, \Delta \qquad (2.79)$$

TABLE - 2.1

Comparison between exact and approximate values of λ^2 for the first mode of vibration of fixed-fixed beam.

K	Δ	Values of λ^2 from exact and Approximate Analyses					
		$\gamma = 4.0$		$\gamma = 8.0$		$\gamma = 12.0$	
		Exact	Approximate*	Exact	Approximate*	Exact	Approximate
1.0	0.0	4.9070 579.792	4.9424 596.679	5.2711 771.995	5.2995 788.746	5.7484 1091.892	5.7702 1108.556
	2.0	4.8026 531.986	4.8296 544.041	5.1872 723.985	5.2088 736.136	5.6840 1043.794	5.7017 1056.895
	4.0	4.4382 387.994	4.4328 386.127	4.9074 579.974	4.9038 578.268	5.4772 899.993	5.4746 898.235
10.0	0.0	6.4844 1767.992	6.6017 1899.473	1959.986	2091.587	2279.795	2411.562
	4.0	6.3006 1575.874	6.4107 1688.920	1767.975	1880.895	2087.992	2200.896
	8.0	5.6233 999.896	5.7022 1057.263	1191.982	1249.352	1511.977	1569.365

* Results from Galerkin's Technique, Eq.2.73

TABLE - 2.2

Comparison between exact and approximate values of λ^2 for the first mode of vibration of a fixed simply supported beam.

K	Δ	Values of λ^2 from Exact and Approximate Analyses*					
		$\gamma = 4.0$		$\gamma = 8.0$		$\gamma = 12.0$	
		Exact	Approximate	Exact	Approximate	Exact	Approximate
1.0	0.0	314.265	325.950	506.302	517.894	826.253	837.892
	2.0	268.632	276.602	460.548	468.596	780.735	788.735
	4.0	132.226	128.557	324.676	320.624	644.378	640.655
10.0	0.0	1439.762	1547.319	1631.753	1739.526	1951.865	2059.296
	4.0	1257.879	1349.926	1449.536	1541.898	1769.758	1861.886
	8.0	712.010	757.747	904.893	949.692	1224.926	1269.686

* Results from Galerkin's Technique, Eq.2.76

3) Beam fixed at one end and Simply Supported at the other: From Equation (2.76) the limit~~f~~ing condition for this case can be obtained as

$$\gamma = 0.559\, n\pi \Delta \qquad (2.80)$$

For the above relations in various cases between γ and Δ, it is really interesting to note that there will be no influence of these two effects on the torsional frequency of vibration. This is because of the opposite nature of their individual effects and these individual effects get nullified at these limit~~f~~ing conditions for various cases.

2.10. REMARKS:

It must be recalled here that the analysis presented in this chapter neglects the effects of longitudinal inertia and shear deformation which are of importance if the effects of cross sectional dimensions on frequencies of vibration are desired. Hence, this analysis is valid for lengthy beams, ie., for beams whose cross sectional dimensions are quite small compared to the length. These second order effects such as longitudinal inertia and shear deformation, therefore, profoundly influence, the frequencies of torsional vibration at higher modes and the propagation of short wave length waves. These effects are taken into consideration in the analyses presented in the ~~coming~~ following chapters.

CHAPTER - III

FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS AND STABILITY OF LENGTHY THIN-WALLED BEAMS ON ELASTIC FOUNDATION*.

3.1. INTRODUCTION:

In Chapter II the title problem is fully analyzed from a purely mathematical approach. This approach provided us with exact solutions for the problem. One short-coming of such an approach is that due to the complex nature of the equation of motion such mathematical difficulties as non-uniform members, complex loadings, or arbitrary boundary conditions can not be easily handled.

To complement the exact solutions given in the previous Chapter, this Chapter intends to provide a means of obtaining approximate solutions to our present problem. The technique used to obtain the approximate results is the method of ''finite'' or ''discrete'' elements. Basically, the finite element method is an extension of the well known Rayleigh-Ritz method in which assumed displacement patterns are specified for an entire structure. In the finite element technique, the continuous system is replaced by a substitute system consisting of a number of finite elements linked together. Once the properties: stiffness, mass and loading of the individual elements have been defined, the equilibrium of the substitute system can be described by a large number of equations, readily solvable on a digital computer.

---

* Part of the results from this Chapter were published by the author, B.V.R.Gupta and D.L.N.Rao in the Proceedings of the International Conference on Finite Element Methods in Engineering, held at Coimbatore Institute of Technology, Coimbatore, India, during 6-7 December 1974. See Ref.(48).

Many of the early advances in the finite element method were presented in technical Journals, but recently two texts have appeared that summarized this modern technique (93, 115). These texts cover such varied topics as plane stress, plane strain, axisymmetric stress analysis, three-dimensional stress analysis, bending of beams and structural stability. To date the finite element method has been used to predict the buckling loads of trusses, beams, plates and shells. In applying the finite element method to these problems in elastic stability it has become necessary to derive the so-called ''initial stress'' or ''stability coefficient'' matrices that account for the in-plane stresses due to in-plane loads.

For problems involving large displacements the stability coefficient matrix has been termed as the ''geometric stiffness'' matrix since it accounts for the influence of large displacements on the equations of equilibrium. Using the conventional elastic stiffness matrix that accounts for the elastic bending stresses, the stability coefficient matrix for small displacements, and the mass matrix that accounts for the inertial loads, a matrix eigenvalue problem is established from which the natural frequencies, critical loads and mode shapes can be determined.

Many investigators used the above technique to predict the buckling loads of trusses (99), beams (68), plates and shells (64,89). Very recently, Pardoen (90) analyzed static and dynamic buckling of thin-walled columns using finite elements and, Barsoum (6) presented a finite element formulation for the general stability analysis of thin-walled members. The method has yet to be extended to the analysis of torsional vibrations and stability of lengthy and short thin-walled beams of open section resting on continuous winkler type elastic foundation.

Thus, a primary objective of this Chapter is to develop, for a lengthy thin-walled beam resting on Winkler type elastic foundation and subjected to an axial time-invariant compressive load, the appropriate stiffness, stability coefficient and, mass matrices necessary for a discrete element torsional vibration and stability analysis. Further, to establish the reliability of the method, the approximate finite element results will be compared with the exact solutions obtained Chapter II.

3.2. FINITE ELEMENT CONCEPT:

The use of finite elements to solve complex problems in structural mechanics has been well documented (115). The method has gained acceptance not only because of its versatility in handling complex structural problems, but also because of the highly systematic manner in which the problem is formulated and subsequently solved. Essentially, the finite element method consists of replacing the actual continuum by a mathematical model composed of structural elements of finite size having known elastic and inertial properties. These structural elements serve as building blocks of the system which, when assembled, provide approximations to the static and dynamic properties of the actual system.

The basic approach in analyzing a thin-walled beam as a net work of discrete elements can be summarized in four steps (26) as follows:

(1) The continuum must be separated by a series of lines or surfaces into a number of ''finite elements''. For a prismatic thin-walled member such as a thin-walled beam, each finite element is represented by a longitudinal segment of the whole beam.

(2) All elements are assumed to be interconnected at a discrete number of boundaries to atleast one adjacent finite element. At each of the connection boundaries a nodal point is designated. For a thin-walled beam the nodal point at the connection boundary is the shear center with generalized displacements such as translations or rotations at this point comprising the basic unknowns of the problem.

(3) The most important step in formulating the finite element procedure is choosing a function or functions to define uniquely the state of displacements within each finite element in terms of its nodal displacements.

(4) Finally, once the displacement function has been determined for the element in terms of nodal displacements, the strain

state within each element can readily be found. Typically, for elastic materials, a differential relationship exists between the displacement and strain states. The strains, together with the appropriate constitutive relation, establish the stress state within the element, the strain energy, potential energy and kinetic energy can be expressed in terms of its generalized nodal displacements.

3.3. BASIC FINITE ELEMENT THEORY FOR VIBRATIONS:

The finite element formulation of the general structural dynamic response problem results in the Equation (26)

$$\bar{M}\ddot{\bar{R}} + \bar{K}\bar{R} - \bar{S}\bar{R} = \bar{F} \tag{3.1}$$

In Eq.(3.1), $\bar{K}$ is the ''total stiffness matrix'' in which the coefficients $\bar{K}_{ij}$ gives the generalized force developed at point i as the result of unit generalized displacement $\bar{R}_j = 1$ imposed on point j, all other points being restrained to zero displacement. The coefficient $\bar{S}_{ij}$ of the ''total stability coefficient matrix'' $\bar{S}$ represents the external load at coordinate i which results in a generalized displacement $\bar{R}_j = 1$ at point j. The coefficient $\bar{M}_{ij}$ of the ''total mass matrix'' $\bar{M}$ represents the mass inertia load at point i developed by a unit acceleration $\ddot{\bar{R}}_j = 1$ at point j. The matrices $\bar{R}$, $\ddot{\bar{R}}$ and $\bar{F}$ are the generalized displacements, accelerations, and loads respectively.

In the finite element deformation method, the deformations of the structure are assumed to be a function of the generalized displacements. The displacements should be continuous accross boundaries of adjoining elements, continuous over the elements, and satisfy the displacement boundary conditions, but they need not satisfy the Cauchy equilibrium equations.

Using the general procedure of the finite element method, the total structure is devided into a number of elements. These elements are connected at their corner or nodal points. Considering a typical three-dimensional element N, the displacements are given by

$$\bar{u}\ (x,\ y,\ z,\ t) = \bar{A}\ (x,\ y,\ z)\ \bar{R}_N(t) \tag{3.2}$$

where the elements of $\bar{u}$ are components of the displacement vector, $\bar{A}$ is a matrix whose elements are functions of the coordinates x, y, and z, and the elements of $\bar{R}_N$ are the generalized coordinates for the N th element with time-invariant magnitudes. The strains are given in terms of nodal displacements using the strain-displacement relation.

Thus,

$$\bar{\epsilon}\ (x,y,z,t) = \bar{C}\ (x,y,z)\ \bar{R}_N(t) \tag{3.3}$$

where $\bar{C}$ is a matrix giving the strains in terms of the generalized displacements $\bar{R}_N$. Using the stress-strain relation, the strain energy can be obtained.

Thus,

$$\bar{\sigma}\ (x,y,z,t) = \bar{D}\ (x,y,z)\ \bar{\epsilon}\ (s,y,z,t) \tag{3.4}$$

where $\bar{\sigma}$ is a matrix of stresses, and the $\bar{D}$ matrix consists of appropriate material constants.

The strain energy U is then given by

$$U = \frac{1}{2} \int_v \bar{\epsilon}^T \bar{\sigma} \, dv \tag{3.5}$$

where $\bar{\epsilon}^T$ represents the transpose of the strain matrix $\bar{\epsilon}$ and v is the volume of the beam.

Substituting Eqs.(3.3) and (3.4) in Eq.(3.5), the strain energy expression becomes,

$$U = \frac{1}{2} \int_v \bar{R}_N^T \bar{C}^T \bar{D} \bar{C} \bar{R}_N \, dv = \frac{1}{2} \bar{R}_N^T \bar{K}_N \bar{R}_N \tag{3.6}$$

where

$$\bar{K}_N = \int_v \bar{C}^T \bar{D} \bar{C} \, dv , \tag{3.7}$$

and is called stiffness matrix for the N th element. Similarly the potential energy can also be written in terms of the generalized coordinates and the stability coefficient matrix $\bar{S}_N$ for the N th element can be obtained.

The kinetic energy T is given by

$$T = \frac{1}{2} \int_v \rho \dot{\bar{u}}^T \dot{\bar{u}} \, dv \tag{3.8}$$

Substituting Eq.(3.2) into Eq.(3.8) we obtain,

$$T = \frac{1}{2} \int_v \rho \dot{\bar{R}}_N^T \bar{A}^T \bar{A} \dot{\bar{R}}_N \, dv = \frac{1}{2} \dot{\bar{R}}_N^T \bar{M}_N \dot{\bar{R}}_N , \tag{3.9}$$

where

$$\bar{M}_N = \int_v \rho \bar{A}^T \bar{A} \, dv. \tag{3.10}$$

and is called the mass matrix for the Nth element. The stiffness, stability coefficient and mass matrices for the complete connected structure is obtained by addition of the component matrices. A given column of the matrix consists of a list of generalized forces at each of the nodes for unit generalized displacement of a given node. When two or more elements have a common node, forces are simply added. Thus if $\bar{K}$ is the final stiffness matrix for the whole structure, the elements of $\bar{K}$ are built as

$$\bar{K}_{ij} = \Sigma\ (\bar{K}_{ij})_N,\ N = 1,2,\ldots \tag{3.11}$$

and similarly

$$\bar{S}_{ij} = \Sigma\ (\bar{S}_{ij})_N\ ,\ N = 1,2,\ldots \tag{3.12}$$

$$\bar{M}_{ij} = \Sigma\ (\bar{M}_{ij})_N\ ,\ N = 1,2,\ldots \tag{3.13}$$

Assuming that the displacements undergo harmonic oscillation, then the displacement vector $\bar{R}_N$ can be written as

$$R_N(t) = \bar{r}_N\ e^{ip_n t} \tag{3.14}$$

where $\bar{r}_N$ is a column vector of amplitudes of the generalized displacements $\bar{R}_N$ and p_n is the circular frequency of oscillation. Substituting Eq.(3.14) into Eq.(3.1) gives:

$$[\bar{K} - \bar{S}]\ [\bar{r}_N] = p_n^2\ [\bar{M}]\ [\bar{r}_N] \tag{3.15}$$

Eq.(3.15) represents an algebraic eigenvalue problem. In this finite element method, the matrices $[\bar{K}]$, $[\bar{S}]$ and $[\bar{M}]$ will be usually symmetric. If the matrices are both symmetric and positive definite, all eigenvalues p_n^2, will be real, positive numbers.

Moreover, the eigen vectors of symmetric matrices are independent; therefore, the matrix $[\bar{r}_N]$ is nonsingular. Another useful property of symmetric matrices is that if the eigenvectors are normalized in such a way that $\{\bar{r}_i\}^T \{\bar{r}_j\} = 1$, the inverse of the modal matrix is equal to the transpose, that is the modal matrix is orthogonal.

The eigenvalue problem for large systems can be solved by numerical schemes that are either direct or iterative. The direct methods are more general and are commonly employed, although the iterative shcemes are suitable for computations when only one or a few of the eigenvalues and their corresponding eigen vectors are needed. Among the various direct approaches to be found in literature are the Jacobi, Givens, Householder and Q R method. Among the iterative techniques are the power or Stodola-Vianello method and inverse iteration. A discussion of these various methods is given in Ref.(111). In the present work, Jacobi's method is utilized in solving the eigenvalue problems.

3.4a. FUNCTIONAL REPRESENTATION OF ANGLE OF TWIST:

In the past the use of polynomials as displacement functions has been popular for describing the displacement within each finite element in terms of its nodal displacements. For the present, to describe the twisting behavior of the thin-walled beam a cubic polynomial is assumed to approximate the angle of twist within each finite element. The motivation for choosing a cubic polynomial is that the contribution to the strain energy due to warping (See Eq.2.2) involves a second derivative of the angle of twist. Choosing a cubic polynomial assures that there will be a non-zero contribution from the warping term whereas if the angle of twist only varied linearly there could be no contribution from the warping term as in this case the second derivative vanishes.

For each finite element of a lengthy thin-walled beam in torsion, there are two generalized nodal displacements at the j end of the ith member. These nodal displacements are:

ϕ_j = angle of twist at the shear center about the longitudinal z-axis;

ϕ'_j = rate of change of angle of twist at the shear center about z-axis;

where the subscript j denotes the generalized displacement at the j end of the ith finite element. Similar generalized nodal displacements exist at the K end of the element. The prime denotes differentiation with respect to z.

If the twist within each finite element is assumed to vary cubicly the displacement function takes the form:

$$\phi(z) = a + bz + cz^2 + dz^3 \qquad (3.16)$$

To establish a relationship between the displacements at any interior coordinate z in terms of the generalized nodal coordinates, the four arbitrary constants in the assumed displacement function must be determined. For instance, the constants a, b, c and d can be determined from the four simultaneous equations given as follows:

$$\phi(0) = \phi_j = a \tag{3.17}$$

$$\frac{\partial\phi}{\partial z}(0) = \frac{\partial\phi_j}{\partial z} = b \tag{3.18}$$

$$\phi(l) = \phi_K = a+bl+cl^2+dl^3 \tag{3.19}$$

$$\frac{\partial\phi}{\partial z}(l) = \frac{\partial\phi_K}{\partial z} = b+2cl+3dl^2 \tag{3.20}$$

where l is the length of the element which is some fraction of the total beam length L.

Once the four coefficients have been determined, the angle of twist at any coordinate z within the element in terms of the four nodal displacements ϕ_j, $\partial\phi_j/\partial z$, ϕ_K and $\partial\phi_K/\partial z$ is uniquely defined, as follows:

$$\phi(z) = \left|(1-3\bar{\xi}_1^2+2\bar{\xi}_1^3),(z-2\bar{\xi}_1 z+\bar{\xi}_1^2 z),(3\bar{\xi}_1^2-2\bar{\xi}_1^3),(-\bar{\xi}_1 z+\bar{\xi}_1^2 z)\right| \left|\begin{matrix}\phi_j\\ \phi'_j\\ \phi_K\\ \phi'_K\end{matrix}\right| \tag{3.21}$$

where $\bar{\xi}_1 = z/l$ is the dimensionless length of the element of the beam.

Eq.(3.6) can be written in an abbreviated form as:

$$\phi(z) = \bar{A}(z)\, \bar{R}_N(t) \qquad (3.22)$$

where

$$\bar{A}(z) = \left[(1-3\bar{\xi}_1^2+2\bar{\xi}_1^3), (z-2\bar{\xi}_1 z+\bar{\xi}_1^2 z), (3\bar{\xi}_1^2-2\bar{\xi}_1^3), (-\bar{\xi}_1 z+\bar{\xi}_1^2 z)\right] \qquad (3.23)$$

and

$$\bar{R}_N = \left[\phi_j\,,\ \phi_j'\,,\ \phi_K\,,\ \phi_K'\right] \qquad (3.24)$$

Similar matrix relations exist for the first and second derivatives of ϕ which can be written as:

$$\phi'(z) = (\bar{A}(z)\, \bar{R}_N(t)\,)' = \bar{A}_1(z)\, \bar{R}_N(t) \qquad (3.25)$$

$$\phi''(z) = (\bar{A}\ (z)\, \bar{R}_N(t)\,)' = \bar{A}_2(z)\, \bar{R}_N(t) \qquad (3.26)$$

where

$$\bar{A}_1(z) = \left[(-6\frac{z}{l^2} + 6\frac{z^2}{l^3}), (1-4\frac{z}{l}+3\frac{z^2}{l^2}), (6\frac{z}{l^2} - 6\frac{z^2}{l^2}), (-2\frac{z}{l} + 3\frac{z^2}{l^2})\right] \qquad (3.27)$$

$$\bar{A}_2(z) = \left[(-\frac{6}{l^2} + 12\frac{z}{l^3}), (-\frac{4}{l} + 6\frac{z}{l^2}), (\frac{6}{l^2} - 12\frac{z}{l^2}), (-\frac{2}{l} + 6\frac{z}{l^2})\right] \qquad (3.28)$$

The generalized velocity and accelerations can also be expressed in terms of the discretized nodal velocities and accelerations. That is:

$$\dot{\phi}(z) = \bar{A}(z)\, \dot{\bar{R}}_N(t) \qquad (3.29)$$

and

$$\ddot{\phi}(z) = \bar{A}(z)\, \ddot{\bar{R}}_N(t) \qquad (3.30)$$

where dots denote differentiation with respect to time t.

3.46 FORMULATION OF ELEMENT MATRICES:

The expressions for the kinetic energy T, strain energy U and potential energy W, derived in Chapter II (See Eqs.2.3, (2.2) and (2.4) respectively) for an element of finite length l can be written as follows:

$$T = \frac{1}{2}\int_0^l \rho I_p (\dot{\phi})^2 dz \qquad (3.31)$$

$$U = \frac{1}{2}\int_0^l \left[EC_w(\phi'')^2 + GC_s(\phi')^2 + K_t(\phi)^2\right] dz \qquad (3.32)$$

and

$$W = \frac{1}{2}\int_0^l \frac{PI_p}{A} (\phi')^2 dz \qquad (3.33)$$

From Hamilton's principle (See Eq.(2.1)) we have:

$$\delta I = \delta \int_{t_1}^{t_2} (T - U + W)\, dt = 0 \qquad (3.34)$$

Direct substitution of Eqs.(3.22), (3.25), (3.26), (3.29) and (3.30) into the energy expressions (3.31), (3.32) and (3.33) yields (for the Nth element):

$$\delta I_N = \delta \int_{t_1}^{t_2} \left\{ \frac{\rho I_p}{2} \int_0^l \dot{\bar{R}}_N^T \bar{A}^T \bar{A} \dot{\bar{R}}_N dz \right.$$

$$- \left[\frac{EC_w}{2} \int_0^l \bar{R}_N^T \bar{A}_2^T \bar{A}_2 \bar{R}_N dz + \frac{GC_s}{2} \int_0^l \bar{R}_N^T \bar{A}_1^T \bar{A}_1 \bar{R}_N dz \right.$$

$$\left. \left. + \frac{K_t}{2} \int_0^l \bar{R}_N^T \bar{A}^T \bar{A} \bar{R}_N dz \right] + \frac{PI_p}{2A} \int_0^l \bar{R}_N^T \bar{A}_1^T \bar{A}_1 \bar{R}_N dz \right\} dt = 0 \qquad (3.35)$$

Eq.(3.35) can be also written more concisely as:

$$\bar{\delta} I_N = \delta \int_{t_1}^{t_2} \frac{1}{2} \Big[(\rho I_p L) \dot{\bar{R}}_{1N}^T \bar{m}_N \dot{\bar{R}}_{1N} - (EC_w/L^3) \bar{R}_{1N}^T \bar{k}_N \bar{R}_{1N}$$

$$+ (PI_p/AL) \bar{R}_{1N}^T \bar{s}_N \bar{R}_{1N} \Big] dt = 0 \tag{3.36}$$

In Eq.(3.36) the terms $(\rho I_p L)\bar{m}_N$, $(EC_w/L^3)\bar{k}_N$ and $(PI_p/AL)\bar{s}_N$ denote respectively the mass matrix $\bar{M}_N$, the stiffness matrix $\bar{K}_N$ and the stability coefficient matrix $\bar{S}_N$ of the Nth element. The matrices $\bar{m}_N$, $\bar{k}_N$, $\bar{s}_N$ and $\bar{R}_{1N}$ are given below:

$$\bar{m}_N = \frac{1}{420N^4} \begin{bmatrix} 156N^2 & & & \\ 22N & 4 & \text{Sym.} & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \tag{3.37}$$

$$\bar{k}_N = \begin{bmatrix} 12N^2 & & & \\ 6N & 4 & \text{Sym.} & \\ -12N^2 & -6N & 12N^2 & \\ 6N & 2 & -6N & 4 \end{bmatrix}$$

$$+ \frac{K^2}{30N^2} \begin{bmatrix} 36N^2 & & & \\ 3N & 4 & \text{Sym.} & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}$$

$$+ \frac{\gamma^2}{105N^4} \left| \begin{matrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{matrix} \right| \tag{3.38}$$

$$\bar{s}_N = \frac{1}{30N^2} \left| \begin{matrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{matrix} \right| \tag{3.39}$$

and

$$\bar{R}_{1N} = |\phi_j,\ L\,\partial\phi_j/\partial Z,\ \phi_K,\ L\,\partial\phi_K/\partial Z| \tag{3.40}$$

where N denotes the number of the elements and $Z = z/L$ is the dimensionless length of the total beam.

The equations of motion for the discretized system can now be obtained by using Eq.(3.36). Taking the variation of the integral expression of Eq.(3.36) we obtain:

$$\int_{t_1}^{t_2} \left[\rho I_p L \delta\dot{\bar{R}}^T_{1N}\, \bar{m}_N\, \dot{\bar{R}}_{1N} - (EC_w/L^3)\delta\bar{R}^T_{1N}\, \bar{k}_N\, \bar{R}_{1N} + (PI_p/AL)\, \delta\bar{R}^T_{1N}\, \bar{s}_N\, \bar{R}_{1N} \right] dt = 0 \tag{3.41}$$

which after integration by parts over the time interval gives:

$$\rho I_p L \delta\bar{R}^T_{1N}\, \bar{m}_N\, \bar{R}_{1N} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta\bar{R}^T_{1N} \left[\rho I_p L\, \bar{m}_N \ddot{\bar{R}}_{1N} + (EC_w/L^3)\bar{k}_N \bar{R}_{1N} - (PI_p/AL)\bar{s}_N\, \bar{R}_{1N} \right] dt = 0 \tag{3.42}$$

The first term in Eq.(3.42) is seen to vanish in view of the assumptions made previously that the virtual displacements $\delta\bar{R}_{1N}$ are zero at the time instants t_1 and t_2. Since the virtual displacement can be arbitrary for other times then the only way in which the integral expression in Eq.(3.42) can vanish is for the terms within the brackets to equal zero. Therefore, the governing dynamic equilibrium equations for the discretized system are:

$$\rho I_p L\, \bar{\bar{m}}_N\, \ddot{\bar{R}}_{1N} + (EC_w/L^3)\, \bar{\bar{k}}_N\, \bar{R}_{1N} - (PI_p/AL)\, \bar{\bar{s}}_N\, \bar{R}_{1N} = \bar{0} \tag{3.43}$$

Assuming that the displacements undergo harmonic oscillation, then the displacement vector $\bar{R}_{1N}$ can be written as:

$$\bar{R}_{1N} = \bar{\bar{r}}_N\, e^{ip_n t} \tag{3.44}$$

where $\bar{r}_N$ is a column vector of torsional amplitudes of the general torsional displacements $\bar{R}_N$ and p_n is the circular frequency of torsional oscillation. Substituting Eq.(3.44) into Eq.(3.43) gives:

$$\left[\left(\frac{EC_w}{L^3}\right)\bar{\bar{k}}_N - \left(\frac{PI_p}{AL}\right)\bar{\bar{s}}_N - \rho I_p L\, p_n^2\, \bar{\bar{m}}_N \right] \bar{r}_N\, e^{ip_n t} = \bar{0} \tag{3.45}$$

Deviding throughout by EC_w/L^3 and cancelling $e^{ip_n t}$, Eq.(3.45) becomes:

$$[\, \bar{\bar{k}}_N - \Delta^2\, \bar{\bar{s}}_N][\bar{r}_N] = \lambda^2[\bar{\bar{m}}_N][\bar{r}_N] \tag{3.46}$$

where Δ^2 and λ^2 are respectively the buckling load and frequency parameters given by:

$$\Delta^2 = \frac{PI_pL^2}{AEC_w} \tag{3.47}$$

and

$$\lambda^2 = \frac{I_pL^4p_n^2}{EC_w} \tag{3.48}$$

Eq.(3.46) represents the equations of motion for an undamped freely oscillating system.

For a beam which is stationary (not vibrating), $\lambda = 0$ and Eq.(3.46) reduces to:

$$[K_N]\ [\bar{r}_N] = \Delta^2\ [\bar{s}_N]\ [\bar{r}_N] \tag{3.49}$$

Eq.(3.49) represents the equations of motion for the torsional buckling of a beam resting on continuous elastic foundation.

3.5. EQUATIONS OF EQUILIBRIUM FOR THE TOTALLY ASSEMBLED BEAM:

As previously mentioned, the matrices $\bar{K}_N$, $\bar{S}_N$, $\bar{M}_N$ and $\bar{r}_N$ pertain only to the Nth finite element and are thus denoted as the element matrices. To obtain the total strain energy, potential energy and Kinetic energy of the beam as an assemblage of N finite elements, the standard finite element procedure is employed. The procedure consists of summing the contributions of each element to form overall stiffness, stability coefficient, mass and displacement matrices which reflect the total energy of the entire beam.

The variation of total energy δI for a thin-walled beam consisting of N finite elements is

$$\bar{\delta} I = \sum_{N=1}^{N} \bar{\delta} I_N = \sum_{N=1}^{N} \frac{1}{2} \int_{t_1}^{t_2} \Big[\rho I_p L \delta\dot{\bar{R}}^T_{1N} \bar{\bar{m}}_N \dot{\bar{R}}_{1N}$$

$$- (EC_w/L^3) \delta\bar{R}^T_{1N} \bar{k}_N \bar{R}_{1N} + (PI_p/AL) \delta\bar{R}^T_{1N} \bar{\bar{s}}_N \bar{R}_{1N} \Big] dt = 0 \qquad (3.50)$$

After summation and integration by parts over the time interval Eq.(3.50) becomes:

$$\rho I_p L \; \delta\bar{R}^T \bar{\bar{m}} \bar{R}_1 \Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \delta\bar{R}^T_1 \Big[\rho I_p L \bar{\bar{m}} \bar{R}_1 + (EC_w/L^3) \bar{k} \bar{R}_1 - (PI_p/AL) \bar{\bar{s}} \bar{R}_1 \Big] dt = 0$$

(3.51)

From Eq.(3.51) the equations of equilibrium for the totally assembled beam can be written as:

$$[\bar{k} - \triangle^2 \bar{\bar{s}}] [\bar{\bar{r}}] = \lambda^2 [\bar{\bar{m}}] [\bar{\bar{r}}] \qquad (3.52)$$

where $\bar{k}$, $\bar{\bar{s}}$, $\bar{\bar{m}}$ and $\bar{\bar{r}}$ denote the totally assembled matrices corresponding to the element matrices $\bar{k}_N$, $\bar{\bar{s}}_N$, $\bar{\bar{m}}_N$ and $\bar{\bar{r}}_N$ defined previously. With the two generalized displacements possible at each node and, with the bar segmented into N elements, the number of degrees of freedom is 2 (N+1).

For a beam which is stationary and not vibrating, $\lambda = 0$ and Eq.(3.52) becomes:

$$[\bar{\bar{K}}]\,[\bar{\bar{r}}] = \triangle^2\,[\bar{\bar{s}}]\,[\bar{\bar{r}}] \qquad (3.53)$$

The formulation of the above matrix equilibrium equations for the totally assembled beam, Eqs.(3.52) and (3.53) include all possible degrees of freedom, both free and restrained. The displacement vector $\bar{\bar{r}}$ of this overall joint equilibrium equations is comprised of both degrees of freedom, the unknowns of the problems and known support displacements or boundary conditions.

3.6. BOUNDARY CONDITIONS:

It should be recalled here that for the present finite element formulation, only two generalized displacements are considered at each node. Hence, to modify the total stiffness, mass and stability coefficient matrices for various combinations of end supports the following boundary conditions are to be utilized:

(a) for a ''simply supported end'', the end of the bar does not rotate but is free to warp and hence,

$$\phi = 0 \qquad (3.54)$$

(b) for a ''clamped end'', the end of the bar is built-in rigidly so that no deformation of the end cross section can take place and we have,

$$\phi = 0 \quad \text{and} \quad \phi' = 0 \qquad (3.55)$$

(c) for a ''free end'' the total matrices are to be used without any modification.

3.7. METHOD OF SOLUTION:

A general computer program is written in Fortran IV to suit the IBM 1130 Computer at the Computer Center, Andhra University, Waltair, in order to obtain the eigenvalues i.e., frequency parameter λ^2 and buckling load parameter $\triangle$ for various values of the foundation parameter γ, and their associated eigen vectors for various end conditions.

The steps involved in the computation program are as follows:

1. To read in the element properties, number of elements N, and boundary conditions.
2. To form element stiffness, stability coefficient and mass matrices.
3. To assemble the total stiffness, stability coefficient and mass matrices.
4. To modify the total matrices according to the specified boundary conditions.
5. To solve the eigenvalue problem utilizing Jacobi's method.
6. To print the given element properties, boundary conditions, number of elements, eigenvalues and their associated eigenvectors.

3.8. RESULTS AND CONCLUSIONS:

The values of λ^2 for the first five frequencies of torsional vibration of simply-supported beam, obtained for a division of the beam into $N = 2,4$ and 6 segments for values of Warping parameter $K = 1$ and 10, and for values of foundation parameter $\gamma = 2,4,6,8,10$ and 12 are shown in Tables 3.1 and 3.2 respectively, which can be observed to compare well with the exact results obtained in Chapter II. The values of λ^2 for the first five torsional frequencies of simply supported beam, for a division of the beam into $N = 6$ segments, for values of warping parameter $K = 0.01$ and 0.1, for various values of $\gamma = 2,4,6,8,10$ and 12 are presented in Tables 3.3 and 3.4 respectively and have compared well with the exact ones.

In Tables 3.5 and 3.6 the results for free-free and fixed-fixed beams are presented respectively for a division of the beam into $N = 6$ segments for values of $K = 0.01$, 0.1, 1.0 and 10 for various values of $\gamma = 2,4,6,8,10$ and 12. From the results presented in Tables 3.1 to 3.6, it can be observed that the frequency parameter λ^2 increases for increasing values of the foundation parameter γ. It can also be observed that as the mode number n increases (ie., for higher modes) the influence of foundation parameter γ decreases. The influence of increasing values of the warping parameter K can be observed to be increasing the frequency parameter λ^2 irrespective of the effect of the continuous elastic foundation. It can be concluded

TABLE - 3.1

Values of the frequency parameter $\bar{\lambda}^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter $\bar{F}$ for a value of warping parameter $\bar{K} = 1$.

$\bar{F}_{\gamma}$	Number of Mode	Number of Elements			Exact Results
		2	4	6	
0	I II III IV V	----	----	107.28663 1600.54248 8041.36524 25687.86724 64403.65635	107.443 1600.56 7991.74 25134.9 61225.6
2	I II III IV V	124.05284 1975.99805 12240.98830 40503.98448 ----	123.32942 1626.36743 8286.18947 30985.92192 77881.84396	123.29409 1616.54785 8057.38282 25703.84770 64419.64073	123.443 1616.56 8007.74 25510.9 61241.6
4	I II III IV V	172.05264 2023.99731 12288.99026 40551.96885 ----	171.32846 1674.36645 8334.18362 30943.92583 77929.82833	171.28823 1664.54345 8105.38184 25751.87114 64467.64854	171.443 1664.56 8055.74 25198.0 61289.6

T A B L E - 3.1 (Contd.)

6	I	252.05285	251.32824	251.29016	251.443
	II	2103.99805	1754.36548	1744.54053	1744.56
	III	12368.98440	8414.18557	8185.38965	8135.74
	IV	40631.98448	31023.93364	25831.85161	25278.9
	V	---	78009.85958	64547.63291	61369.6
8	I	364.05273	363.32800	363.29058	363.443
	II	2215.99805	1866.36572	1856.55054	1856.56
	III	12480.99026	8526.18947	8297.37502	8247.74
	IV	40743.98448	31135.92583	25943.87114	25390.9
	V	---	78121.84396	64659.64854	61481.6
10	I	508.05279	507.32733	507.28814	507.443
	II	2359.99756	2010.36572	2000.54663	2000.56
	III	12624.98635	8670.18947	8441.35939	8397.74
	IV	40887.87666	31279.92583	26087.83208	25534.9
	V	---	78265.84396	64803.60948	61625.6
12	I	684.05285	683.32605	683.28711	683.443
	II	2535.99756	2186.36524	2176.53809	2176.56
	III	12800.98830	8846.18557	8617.34767	8567.74
	IV	41063.97666	31455.92583	26263.88677	25710.9
	V	---	78441.84396	64779.62510	61801.6

TABLE - 3.2

Values of the Frequency parameter $\bar{\lambda}^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter $\bar{F}$ for a value of warping parameter $\bar{K} = 10$.

$\bar{F}_\gamma$	Number of Mode	Number of Elements			Exact Results
		2	4	6	
0	I			1084.37207	1085.32
	II			5509.04395	5512.07
	III	---	---	16838.35552	16792.6
	IV			41347.17977	40780.9
	V			88955.62521	85672.6
2	I	1101.46338	1100.42187	1100.37646	1101.32
	II	5935.99415	5536.11720	5525.04298	5528.07
	III	21571.76177	17105.23832	16854.35161	16808.6
	IV	57135.97666	46735.88291	41363.17196	40796.9
	V	---	102839.67208	88871.62521	85688.6
4	I	1149.46362	1148.42114	1148.37182	1149.32
	II	5983.99513	5584.11913	5573.03223	5576.07
	III	21619.76177	17153.24223	16902.34770	16856.6
	IV	57183.96885	46783.87510	41411.21102	40924.9
	V	---	102887.67208	89019.65646	85736.6

T A B L E - 3.2 (Contd.)

6	I	1229.46362	1228.42090	1228.37012	1229.32
	II	6063.99317	5664.12306	5653.03907	5656.07
	III	21699.76177	17283.23832	16982.35552	16936.6
	IV	57263.96885	46863.88291	41491.14852	40924.9
	V	---	102967.67208	89099.65646	85816.6
8	I	1341.46338	1340.42163	1340.37402	1341.32
	II	6175.99610	5776.11427	5765.04102	5768.07
	III	21811.76177	117345.25004	17094.35552	17048.6
	IV	57375.97666	46975.88291	41603.16415	41036.9
	V	---	103079.73458	89211.59396	85928.6
10	I	1485.46362	1484.41992	1484.37036	1485.32
	II	6319.99513	5920.11915	5909.04298	5912.07
	III	22131.76177	17489.23832	17238.35161	17192.6
	IV	57519.95323	47119.86729	41747.17977	41180.9
	V	---	103223.67208	89355.64083	86072.6
12	I	1661.46338	1660.42016	1660.36767	1661.32
	II	6495.99415	6096.11915	6085.04981	6088.07
	III	22131.75786	17665.23442	17414.33989	17368.6
	IV	57695.98448	47295.87510	41923.15634	41356.6
	V	---	103399.65646	89531.64083	86248.6

TABLE - 3.3

Values of the Frequency parameter $\bar{\lambda}^2$ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter $\bar{r}$ for a value of warping parameter $K = 0.01$.

$\bar{r}$	Number of Mode	Number of Elements 6	Exact Results
0	I	97.42036	---
	II	1561.06689	
	III	7952.49806	
	IV	25529.69145	
	V	64155.57041	
2	I	113.42420	113.566986
	II	1577.08105	1577.060061
	III	7968.49513	7918.854505
	IV	25545.68363	24992.914115
	V	64171.59385	60994.773544
4	I	161.41571	161.566986
	II	1625.07593	1625.060061
	III	8016.49122	7966.854505
	IV	25593.67192	25040.914115
	V	64219.60948	61042.773544
6	I	241.41577	241.566986
	II	1705.07324	1705.060061
	III	8096.49122	8046.854505
	IV	25673.68363	25120.914115
	V	64299.58604	61122.773544
8	I	353.42068	353.567017
	II	1817.07251	1817.060061
	III	8208.49221	8159.854505
	IV	25785.68754	25232.914115
	V	64411.55479	61234.773544
10	I	497.42071	497.567017
	II	1961.07226	1961.060061
	III	8352.50002	8302.855491
	IV	25929.67582	25376.914115
	V	64555.60948	61378.773544
12	I	673.41674	673.567018
	II	2137.07080	2137.060065
	III	8528.49807	8478.855491
	IV	26105.66801	25552.914115
	V	64731.57823	61554.773544

TABLE - 3.4.

Values of the Frequency parameter λ for simply supported thin-walled beams of open section on Elastic foundation for various values of foundation parameter γ for a value of warping parameter $\bar{K} = 0.100$.

γ	Number of Mode	Number of Elements 6	Exact Results
0	I	97.51748	---
	II	1561.46094	
	III	7953.36817	
	IV	25531.24613	
	V	64158.03915	
2	I	113.52183	113.664779
	II	1577.46582	1577.451174
	III	7969.38282	7919.735364
	IV	25547.23442	24994.476615
	V	64174.04696	60997.218841
4	I	161.51513	161.664795
	II	1625.46216	1625.451174
	III	8017.38184	7967.735364
	IV	25595.25395	25042.437615
	V	64222.00010	61045.218841
6	I	241.51611	241.664795
	II	1705.46167	1705.451174
	III	8097.40040	8047.735364
	IV	25675.24613	25122.476615
	V	64302.00790	61125.218841
8	I	353.51928	353.664795
	II	1817.46264	1817.451174
	III	8209.38088	8159.735364
	IV	25787.25786	25234.476615
	V	64413.99220	61237.218841
10	I	497.51690	497.664795
	II	1961.46142	1961.451174
	III	8353.38479	8303.736354
	IV	25931.24613	25378.476615
	V	64558.05477	61381.218841
12	I	673.51562	673.664796
	II	2137.45606	2137.45117
	III	8529.28283	8479.736354
	IV	26107.25395	25554.476615
	V	64734.04696	61457.218841

TABLE - 3.5

Values of the Frequency parameter λ^2 for fixed-fixed thin-walled beams of open section on Elastic foundation for various values of foundation and warping parameters γ and K respectively (N = 6).

K	Mode No	Values of γ						
		0	2	4	6	8	10	12
0.01	I	500.82769	516.82995	564.82458	644.82568	756.82763	900.82885	1076.82617
	II	3818.47852	3834.48145	3882.47363	3962.47803	4074.46875	4218.48048	4394.46583
	III	14827.81642	14843.81642	14891.81642	14971.81642	15083.82424	15227.81642	151403.80861
	IV	41352.17196	41368.14852	41416.12509	41496.10946	41608.14071	41752.14071	41928 .14071
	V	93087.53146	93103.51583	93151.56271	93231.57833	93343.54708	93487.50021	93663.53146
0.10	I	500.94867	516.95300	564.94653	644.94665	756.94885	900.94922	1076.94800
	II	3818.93018	3834.93115	3882.93213	3962.93604	4074.92383	4218.93458	4394.92969
	III	14828.78713	14844.79299	14892.80080	14972.78713	15084.79689	15228.78713	15404.79494
	IV	41353.86729	41369.86729	41417.85948	41497.85166	41609.85166	41753.84385	41929.84385
	V	93090.20330	93106.14080	93154.15643	93234.18768	93346.18768	93490.17205	93666.18768
1.00	I	513.12353	529.12585	577.11975	657.12426	769.12646	913.12133	1089.12060
	II	3864.54248	3880.54199	3928.53418	4008.54492	4120.53712	4262.54102	4440.53419
	III	14927.01760	14943.01955	14991.00978	15071.02936	15183.01955	15327.01369	15503.01174
	IV	41525.59385	41541.58604	41589.58604	41669.60166	41781.58604	41925.60948	42101.59385
	V	93361.31271	93377.31271	93425.34396	93505.29708	93617.31271	93761.28146	93937.28146
10.0	I	1683.98877	1699.98999	1747.98486	1827.98315	1939.98266	2083.98535	2259.98486
	II	8360.31252	8376.32033	8424.31056	8504.31447	8616.31838	8760.31252	8936.31252
	III	24732.19145	24748.21488	24796.19926	24876.21098	24988.19926	25132.20317	25308.20707
	IV	58721.25790	58737.28134	58785.24227	58865.25790	58977.24227	59121.25009	59297.26571
	V	120469.45333	120485.46896	120533.45333	120613.48458	120725.48458	120869.45333	121045.48458

TABLE - 3.6

Values of the Frequency parameter λ^2 for free-free thin-walled beams of open section on Elastic Foundation for various values of foundation and warping parameters γ and K respectively (N = 6).

K	Mode No	Values of γ						
		0	2	4	6	8	10	12
0.010	I	0.00114	16.00515	63.99741	143.99972	256.00122	399.99932	576.0036
	II	0.00815	16.01287	64.00315	144.01077	257.01220	400.00708	576.01135
	III	500.80816	516.81799	564.80713	644.82763	756.81396	900.81250	1076.81518
	IV	3816.42822	3832.41455	3880.41085	3960.41699	4072.41992	4216.42481	4392.42090
	V	14785.43752	14801.44143	14849.44143	14929.45510	15041.43752	15185.45705	15361.43752
0.100	I	0.00094	16.00503	64.00390	144.00061	256.00116	399.99908	576.00036
	II	0.12253	16.12235	64.12205	144.11544	256.12005	400.13055	576.11731
	III	501.30493	517.30456	565.31860	645.31494	757.29931	901.31140	1077.29736
	IV	3817.47998	3833.48096	3881.46875	3961.50342	4073.48340	4217.49610	4393.47559
	V	14787.30080	14803.31056	14851.32434	14931.32814	15043.29689	15187.32033	15363.29885
1.00	I	0.00117	16.00636	63.09979	143.99942	255.99951	399.99969	575.99572
	II	11.95039	27.94924	75.95919	155.94809	257.95068	411.95587	587.94665
	III	550.23401	566.23986	614.22033	694.23193	806.22277	950.21899	1126.19848
	IV	3925.66797	3941.63721	3989.68360	4069.65088	4131.66602	4325.68653	4501.66407
	V	14974.07033	14990.07033	15038.07033	15118.06642	15230.08791	15374.05471	15550.04103
10.0	I	0.00858	16.00782	64.00463	144.00341	256.00756	400.00787	576.00903
	II	1046.32202	1062.31982	1110.32690	1190.31909	1302.32104	1446.32300	1622.32055
	III	4926.61915	4942.59766	4990.61915	5070.60743	5132.61524	5326.61915	5502.62013
	IV	14068.30471	14084.29689	14132.31252	14212.30471	14324.32228	14468.32619	14644.31056
	V	33131.40634	33147.40634	33195.40634	33275.39071	33337.41415	33531.39071	33707.40634

therefore, that increase in the values of warping parameter K and foundation parameter γ contribute for the increase in the torsional frequency parameter λ^2.

In Tables 3.7, 3.8 and 3.9, the values of the frequency parameter λ^2 for the first five modes of vibration are presented for simply-supported, fixed-fixed and, fixed-simply supported beams respectively, for various values of axial load parameter Δ and foundation parameter γ, for a value of warping parameter K = 1. These results are given for a division of the beam into four and six segments. It can be observed from Table 3.7, that the results for the simply-supported beams compare well with the exact ones. It can be also noticed that increase in the value of axial load parameter Δ, for any constant or zero values of the foundation parameter γ and warping parameter K, is to decrease the value of the frequency parameter λ^2. Similarly it can be observed that, for any constant or zero values of the axial load parameter Δ, the increase in the values of foundation parameter γ and warping parameter K is to increase the value of the frequency parameter λ^2.

Hence It can be concluded that the combined influence of axial load parameter Δ, foundation parameter γ and warping parameter K on the frequency parameter λ^2 is the algebraic sum of the individual influences of these parameters. In general, for all the cases presented here, the results from the finite element analysis are in excellent agreement with the exact results from Chapter II, and the convergence of the results is quite satis-

T A B L E - 3.7

Values of the frequency parameter λ^2 for simply supported beams for various values of axial load parameter $\triangle$ and foundation parameter γ for a value of K = 1.

Value of γ	Values of $\triangle$	No.of Mode	Number of Elements		Exact Results
			4	6	
0.0	2.0	I	67.5211	67.6339	67.8010
		II	1286.9080	1294.5712	1440.1235
		III	7940.3071	7678.1423	7623.7256
		IV	30635.5009	25771.8011	24463.2071
		V	76947.9860	63627.9682	60141.0001
6.0	2.0	I	211.5192	21.9334	211.8011
		II	1430.9046	1438.5728	1584.1235
		III	8084.3116	7822.1494	7767.7256
		IV	30779.5448	25915.8241	24607.2071
		V	77092.2992	63772.0157	60285.0001
6.0	3.5	I	129.9805	117.5148	130.3764
		II	1107.5051	1111.9768	1258.4253
		III	7348.5527	7088.6543	7034.9043
		IV	29453.4244	24607.4609	23304.4141
		V	75010.2544		58249.3829

T A B L E - 3.8

Values of the frequency parameter λ^2 for fixed-fixed beams for various values of axial load parameter $\triangle$ and foundation parameter γ for a value of K = 1.

Value of γ	Value of $\triangle$	Mode No.	Number of Elements	
			4	6
0.0	2.0	I	606.6059	474.5637
		II	3732.2152	3719.4751
		III	14463.6348	14153.3851
		IV	53954.4631	40699.0235
		V	146916.5902	93660.2189
6.0	2.0	I	750.6064	625.8259
		II	3876.2219	3863.4802
		III	14607.4632	14297.6426
		IV	54098.5733	40843.0235
		V	147060.7452	93810.2189
6.0	6.6	I	266.1976	210.3856
		II	2036.6875	2022.8406
		III	10659.9258	10340.2461
		IV	46728.1719	33990.2423
		V	135125.3488	82926.8439

TABLE - 3.9

Values of the frequency parameter λ^2 for fixed-simply supported beams for various values of axial load parameter Δ and foundation parameter γ for a value of K = 1.

Value of γ	Value of Δ	Mode No.	Number of Elements	
			4	6
0.0	3.0	I	178.7215	149.8247
		II	2069.1348	1931.5761
		III	10244.8752	10065.7076
		IV	38389.3486	30721.6524
		V	102718.9681	76119.4065
6.0	3.0	I	322.7196	299.8219
		II	2213.1309	2075.5769
		III	10388.8594	10209.6878
		IV	38533.2974	30865.6524
		V	102863.2427	76263.4221
6.0	4.7	I	257.3521	210.5931
		II	1653.0818	1498.0771
		III	9167.3867	9004.8523
		IV	36271.5834	28703.0196
		V	99361.3712	72872.9669

factory for a division of the beam into six elements. Hence, the finite element model presented in this Chapter, which includes the effects of warping, axial compressive load and elastic foundation is quite satisfactory and yields good results.

CHAPTER - IV.

EFFECT OF LONGITUDINAL INERTIA AND OF SHEAR DEFORMATION ON THE TORSIONAL FREQUENCIES AND NORMAL MODES OF SHORT WIDE-FLANGED THIN-WALLED BEAMS OF OPEN SECTION.*

4.1. INTRODUCTION:

In the analytical studies presented in Chapters II and III, the problems are formulated utilizing the Timoshenko torsion theory (98) and, the effects of longitudinal inertial and shear deformation are neglected assuming the beam to be lengthy compared to the cross sectional dimensions. But the corrections due to longitudinal inertia and shear deformation may be of importance if the effects of cross sectional dimensions on the frequencies of torsional vibration are desired.

Timoshenko torsion theory, though intended to be an improvement over the classical Saint-Venant torsion theory, suffers from the defect that while dispersive in character, very short wavelengths are propagated with infinite velocities. Thus, this improved theory is limited in its description of high-frequency (short-wavelength) vibrations and, because it contains no delay time (infinite velocities), it is not suited for problems involving the response to sharp transients. So ~~much so~~, Timoshenko torsion theory cannot be justified for short wide-flanged beams and higher modes of vibration.

---

* Results from this Chapter were published by the author, K.V.Apparao and P.K.Sarma in May, 1974 issue of the Journal of the Aeronautical Society of India, see Ref.(49).

Though there exists some studies 3,4,70,104 (~~1,2,3~~) on free torsional vibrations of beams of open section including second order effects such as longitudinal inertia, shear deformation and shear lag, solutions were given only for the simple case of a simply supported beam. Stating that the frequency equations for other boundary conditions are highly transcendental in nature, their solutions were not attempted. The effects of longitudinal inertia and shear deformation on torsional frequencies for various boundary conditions of short wide-flanged thin-walled beams of open section were not yet fully analyzed. Further, it is observed that the torsional frequency values for Indian standard wide-flanged I-beams are not ~~made~~ available until now in the literature, ~~till~~ , ~~now.~~

The present chapter therefore deals with exact and approximate analytical solutions of torsional vibrations of short wide-flanged thin-walled beams of open section, for which the shear center and centroid coincide, including the effects of longitudinal inertia and shear deformation. The governing equations of motion are desired using Hamilton's principle. The method of solution used by Huang (69) in the analysis of Timoshenko beam equations in flexural vibrations, is applied to the coupled equations of motion to derive a clear and neat set of frequency and normal mode equations for six common types of simple and finite beams. Solutions are obtained for two complete differential equations in angle of twist and warping angle respectively.

The constants in these solutions are related by any one of the original coupled equations from which the two complete equations are derived. The advantage of this method is that the boundary conditions prescribed are homogeneous and the analysis becomes quite simple. The expressions for orthogonality and normalizing conditions for the principal normal modes, which are useful in solving forced vibration problems and, which include both the angle of twist and warping angle are also obtained in this Chapter for both the general case and for beams with various simple end conditions.

To facilitate can by the designers, extensive design data ~~is~~ are presented for the torsional frequencies of Wide-flanged doubly symmetric I-beams with various types of end conditions. The results for the first four modes of vibration for various types of end conditions are presented in tabular form suitable for design use.

To supplement the exact solutions, with approximate analytical solutions, the problem is also solved for some typical boundary conditions utilizing the Galerkin's technique. Depending upon the assumed functions satisfying the prescribed boundary conditions of the beam, Galerkin's technique is found to give nearly accurate results.

4.2. BASIC ASSUMPTIONS:

The problems investigated in this Chapter are restricted to the following assumptions:

a) The material of the beam is homogeneous, isotropic and obeys Hooke's law.

b) By symmetry, the cross sections rotate with respect to centroidal axis, the warping is confined to flanges only.

c) Plane cross sections of different straight pieces remain plane, and warping accross the thickness of these cross sections is neglected.

d) The distortion of the wab out of its plane is assumed negligible.

e) Bending of the flanges does not produce any additional shear stresses on the flange-web section.

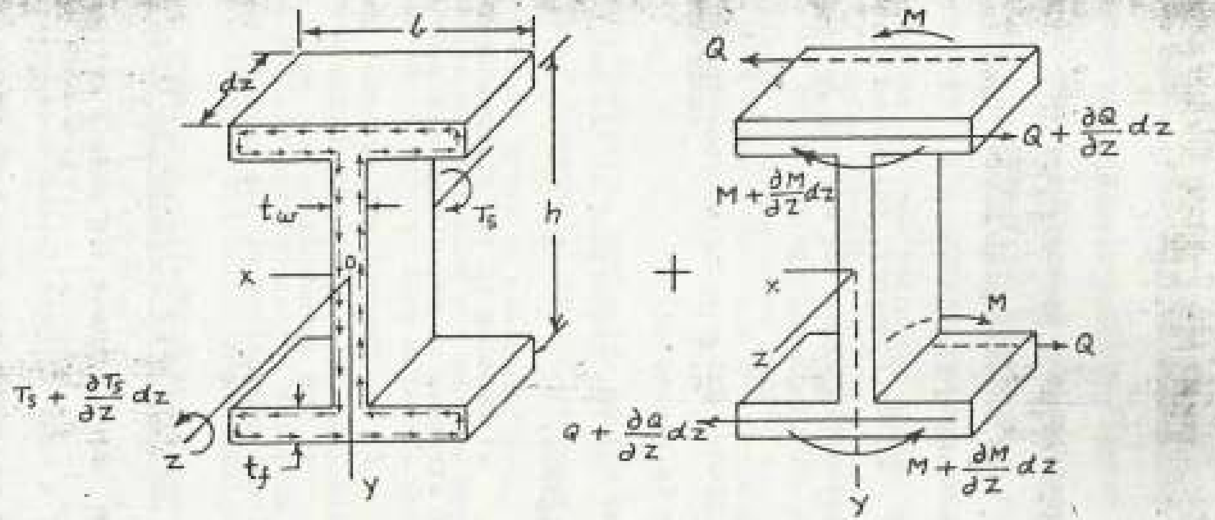
f) No internal and external damping forces exist.

g) The deformations are small compared with the cross-sectional dimensions of the beam in the linearized problem.

4.3. DERIVATION OF DIFFERENTIAL EQUATIONS OF MOTION:

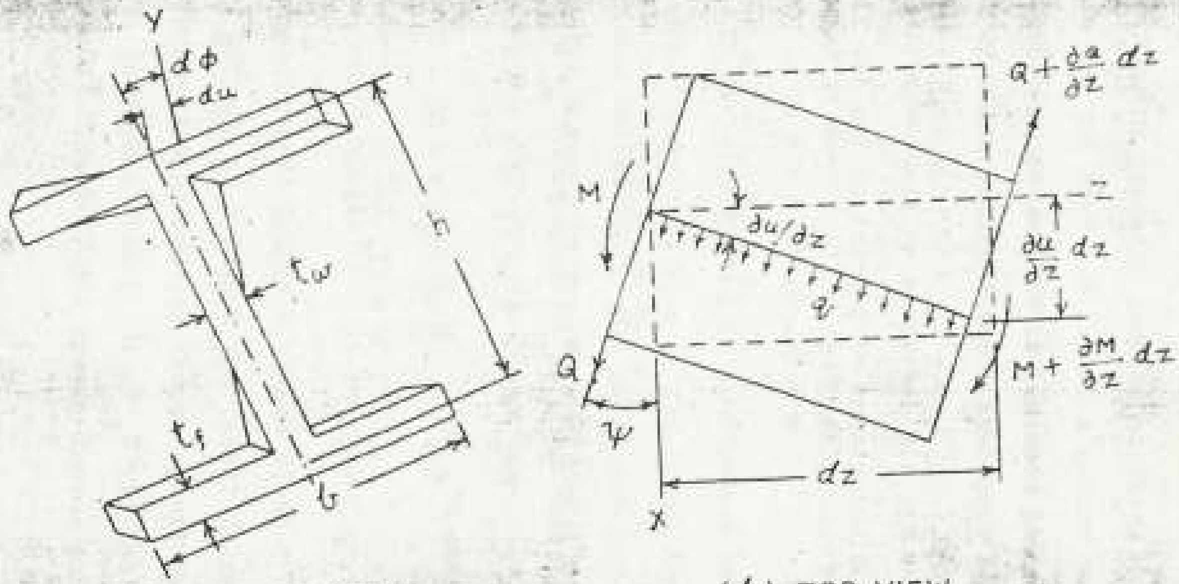
Figs.4.1 and 4.2 show a differential element of length dz of a wide-flanged I-beam undergoing torsion. The strain energy U_1 at any instant t in a beam of length L due to Saint-Venant torsion is (See Eq. 2.2a)

$$U_1 = \frac{1}{2}\int_0^L GC_s\left(\frac{\partial\phi}{\partial z}\right)^2 dz \qquad (4.1)$$


l
dz
t_w
T_s
h
x
o
$T_s + \frac{\partial T_s}{\partial z} dz$
z
t_f
y
+
M
Q
$Q + \frac{\partial Q}{\partial z} dz$
$M + \frac{\partial M}{\partial z} dz$
x
M
Q
z
$Q + \frac{\partial Q}{\partial z} dz$
$M + \frac{\partial M}{\partial z} dz$
y

(a) (b)

FIG. 4.1 - GEOMETRY AND FORCES ON A DIFFERENTIAL ELEMENT I SECTION


Y
$d\phi$
du
h
t_w
t_f
b
M
$\partial u/\partial z$
q
Q
ψ
$Q + \frac{\partial Q}{\partial z} dz$
z
$\frac{\partial u}{\partial z} dz$
$M + \frac{\partial M}{\partial z} dz$
dz
x

(a) FRONT VIEW

(b) TOP VIEW

FIG.4.2- STRAINED STATE OF A BEAM ELEMENT

Accompanying the rotation is a warping of the cross-section which is assumed constant in each piece of the cross-section having a moment M. Thus for the wide-flanged section, warping is confined to flanges alone and its angle of rotation denoted by ψ (z,t); see Figs.4.1 and 4.2.

Fig.4.2 (b) shows an element of the top flange. If w is the z-displacement of a point in the top flange, then

$$w = (x, z, t) = - x \psi \qquad (4.2)$$

and the z-component of strain is given by

$$\epsilon_z = \frac{\partial w}{\partial z} = - x \frac{\partial \psi}{\partial z} \qquad (4.3)$$

The section is thin, so we assume $\sigma_x = \sigma_y = 0$, and Hooke's law gives $\sigma_z = E\epsilon_z$, where E is Young's modulus. Moment M due to stresses σ_z is

$$M = EI_f \frac{\partial \psi}{\partial z} \qquad (4.4)$$

It is easily verified that stresses σ_z give rise to no net axial force, and moment M in the top flange and -M in the bottom flange cancel so that no net moment M_y exists on the cross-section. If U_2 is the strain energy of the two flanges due to the warping normal strain (98), then

$$U_2 = \frac{1}{2} \int_0^L 2M\left(\frac{\partial \psi}{\partial z}\right)dz = \frac{1}{2} \int_0^L 2EI_f\left(\frac{\partial \psi}{\partial z}\right)^2 dz \qquad (4.5)$$

If ϵ_{sh} is the shear strain at the center of the flange,

$x = 0$, then by definition

$$\varepsilon_{sh} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} - \psi \tag{4.6}$$

where u is the x-displacement of the top flange center line. Eq.(4.6) introduces the effect of transverse shear deformation used for bars by Timoshenko (10) and later applied to plates (7). Using Hooke's law for shear, the value of ε_{sh} given by Eq.(4.6) is assumed proportional to the total shear force Q,

$$- Q = K' A_f G \varepsilon_{sh} \tag{4.7}$$

where A_f is the cross sectional area of the flange, and K' is the transverse shear coefficient. The equal and opposite shear forces Q, a distance h apart in the top and bottom flanges, give rise to a torque due to warping, T_w, given by

$$T_w = - Qh = K' A_f Gh\left(\frac{h}{2}\frac{\partial \phi}{\partial z} - \psi\right) \tag{4.8}$$

in which displacement compatibility at the web-flange joint

$$u = (h/2)\,\phi \tag{4.9}$$

has been used to eliminate u in Eq.(4.6).

The total torsional couple, T_t, on the cross section is given from Eqs.(2.2a) and (4.8) as

$$T_t = T_s + T_w = GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh\left(\frac{h}{2}\frac{\partial \phi}{\partial z} - \psi\right) \tag{4.10}$$

The strain energy due to shear deformation of the two flanges, U_3, is

$$U_3 = \frac{1}{2}\int_0^L 2(-Q)\,c_{sh}\,dz = \frac{1}{2}\int_0^L 2K'A_fG\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)^2 dz \qquad (4.11)$$

The total strain energy, U, at any instant t is given from Eqs.(4.1), (4.5) and (4.11) by

$$U = U_1 + U_2 + U_3 = \frac{1}{2}\int_0^L \left[GC_s\left(\frac{\partial\phi}{\partial z}\right)^2 + 2EI_f\left(\frac{\partial\psi}{\partial z}\right)^2 + 2K'A_fG\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)^2\right] dz \qquad (4.12)$$

The total kinetic energy at time t is

$$T_k = \frac{1}{2}\int_0^L \left[\rho I_p\left(\frac{\partial\phi}{\partial t}\right)^2 + 2\rho I_f\left(\frac{\partial\psi}{\partial t}\right)^2\right] dz \qquad (4.13)$$

where the first term is the Kinetic energy of torsional rotation ϕ and the second term is that due to longitudinal (warping) displacements of the two flanges.

Since our object here is to study the free vibrations of the beam, the potential energy, W, of the external force system is taken as zero. If T_k and U from Eqs.(4.12) and (4.13) are substituted into the Hamilton integral given by Eq.(2.1), and variations taken, and after integrating the first two terms by parts with respect to t and next three with respect to z, we obtain:

$$\int_{t_0}^{t_1}\int_0^L \left[\left\{GC_s\frac{\partial^2\phi}{\partial z^2} + K'A_fGh\left(\frac{h}{2}\frac{\partial^2\phi}{\partial z^2} - \frac{\partial\psi}{\partial z}\right) - \rho I_p\frac{\partial^2\phi}{\partial t^2}\right\}\delta\phi\right.$$

$$\left. + \left\{2EI_f\frac{\partial^2\psi}{\partial z^2} - 2\rho I_f\frac{\partial^2\psi}{\partial t^2} + 2K'A_fG\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)\right\}\bar{\delta}\psi\right] dz\,dt$$

$$+ \int_0^L \left(\rho I_p\frac{\partial\phi}{\partial t}\delta\phi + 2\rho I_f\frac{\partial}{\partial t}\bar{\delta}\psi\right)\Bigg|_{t_0}^{t_1} dz$$

$$- \int_{t_0}^{t_1} \left[\left\{ GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right\} \bar{\delta}\phi + 2EI_f \frac{\partial \psi}{\partial z} \bar{\delta}\psi \right]_0^L dt = 0$$

(4.14)

Assuming that the values of ϕ and ψ are given at the two fixed instants, the second integral vanishes. If the boundary conditions are such that the third integral also vanishes, then we obtain the following two coupled equations of motion:

$$GC_s \frac{\partial^2 \phi}{\partial z^2} + K' A_f Gh \left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z} \right) - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.15)$$

and

$$EI_f \frac{\partial^2 \psi}{\partial z^2} + K' A_f G \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) - \rho I_f \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad (4.16)$$

4.4.(a) NATURAL BOUNDARY CONDITIONS:

In deriving the coupled equations (4.15) and (4.16) from (4.14) it was assumed that the expression

$$\left[GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right] \delta\phi + 2EI_f \frac{\partial \psi}{\partial z} \bar{\delta}\psi$$

vanishes at the ends $z=0$ and $z=L$. This condition is satisfied if at the two ends,

$$\left[GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right] \delta\phi = 0, \qquad (4.17)$$

and

$$\frac{\partial \psi}{\partial z} \bar{\delta}\psi = 0. \qquad (4.18)$$

Eqns.(4.17) and (4.18) give the natural boundary conditions for the finite bar, and are satisfied if the end conditions are taken as:

1. $\phi = 0$ and $\frac{\partial \psi}{\partial z} = 0$ (4.19)

These conditions imply no end rotation and zero bending moment in the flange-ends. In this case, the web is constrained against rotation while the flanges are free to warp. This is the case of a ''Simply Supported end''.

2. $\phi = 0$ and $\psi = 0$ (4.20)

These conditions imply constraint against end rotation as well as end warping, and hence give no end deformation. These conditions define a ''built-in end''.

3. $\frac{\partial \psi}{\partial z} = 0$ and $GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) = 0$ (4.21)

These conditions imply zero bending moment in the flange ends and no torque at the end cross section. The end is thus free from tractions and the conditions correspond to a ''free end''.

4. $\psi = 0, \quad GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) = 0$

or equivalently,

$$\psi = 0, \quad \frac{\partial \phi}{\partial z} = 0 \qquad (4.22)$$

The latter conditions imply no warping and zero shear forces in the end flanges.

These conditions are useful for finding symmetric modes of vibration in simply supported, fixed-fixed, and free-free beams.

(b) TIME-DEPENDENT BOUNDARY CONDITIONS:

The homogeneous boundary conditions discussed above give the free vibrations of beams. For forced vibrations produced by the motion of boundaries, appropriate time dependent end conditions are given by prescribing at each end one member of each of the products:

$$\left[GC_s \frac{\partial \phi}{\partial z} + K' A_f G h \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right] \bar{\delta} \phi \text{ and } EI_f \frac{\partial \psi}{\partial z} \bar{\delta} \psi .$$

or equivalently of:

$$T_t \bar{\delta} \phi \text{ and } M \bar{\delta} \psi .$$

Of the many conditions thus obtained, the following are of more theoretical interest;

1. torque T_t prescribed, bending moment $M = 0$ or $\psi = 0$,
2. ϕ or $\frac{\partial \phi}{\partial t}$ prescribed, bending moment $M = 0$ or $\psi = 0$,
3. bending moment M prescribed, torque $T_t = 0$ or $\phi = 0$,
4. ψ or $\frac{\partial \psi}{\partial t}$ prescribed, torque T_t or $\phi = 0$.

In the case of semi-infinite beams, conditions need be prescribed at one end since all physical quantities at any instant are zero at the far end.

4.5.1. SINGLE EQUATION IN ANGLE OF TWIST:

Eliminating ψ between the coupled equations (4.15)and (4.16), a single equation of motion in angle of twist ϕ may be obtained as:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w\right]\frac{\partial^4\phi}{\partial z^4} - \left[\frac{E\rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right]\frac{\partial^4\phi}{\partial z^2 \partial t^2}$$

$$- GC_s \frac{\partial^2\phi}{\partial z^2} + \rho I_p \frac{\partial^2\phi}{\partial t^2} + \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4\phi}{\partial t^4} = 0 \qquad (4.23)$$

Eq.(4.23) is a linear partial differential equation of fourth order, and is of the same form as the Timoshenko beam equation for flexural vibrations (10), under an axial load P which introduces an additional term $- P \frac{\partial^2 y}{\partial z^2}$ (as spring restoring force) in the Timoshenko equation. It is clear that the term $- GC_s \frac{\partial^2\phi}{\partial z^2}$ is analogous to the term $- P\frac{\partial^2 y}{\partial z^2}$.

4.5.2. ANALYSIS OF VARIOUS TERMS:

i) Letting $C_w = \rho I_f = 0$ and $K' \to \infty$, Eq.(4.23) reduces to:

$$GC_s \frac{\partial^2\phi}{\partial z^2} - \rho I_p \frac{\partial^2\phi}{\partial t^2} = 0 \qquad (4.24)$$

This equation represents Saint Venant torsion theory for slender beams and does not include warping of the cross section, shear deformation and longitudinal inertia effects. It is given in Love (76) and is discussed by Gere (32).

ii) $C_w = 0$ and $K' \to \infty$, then Eq.(4.23) becomes:

$$GC_s \frac{\partial^2\phi}{\partial z^2} + \frac{\rho I_f h^2}{2} \frac{\partial^4\phi}{\partial z^2 \partial t^2} - \rho I_p \frac{\partial^2\phi}{\partial t^2} = 0 \qquad (4.25)$$

The second term represents Love's corrections(76) for the longitudinal inertia added to Eq.(4.24) and corresponds to Rayleigh's correction(100), for lateral inertia in the elementary theory for longitudinal vibrations.

iii) If $\rho I_f = 0$ and $K' \rightarrow \infty$, Eq.(4.23) reduces to:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.26)$$

This equation represents Timoshenko's torsion theory which includes the effect of warping of the cross-section and has been treated in detail by Gere(32).

iv) If $K' \rightarrow \infty$, Eq.(4.23) reduces to:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - \frac{\rho I_f h^2}{2} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.27)$$

This equation represents Love's correction added to Timoshenko's torsion theory and corresponds to Rayleigh's correction of rotary inertia(100), in the Bernoulli-Euler beam theory.

v) If $\rho I_f = 0$, then Eq.(4.23) is given as:

$$\left(\frac{EI_f C_s}{K' A_f G} + EC_w\right) \frac{\partial^4 \phi}{\partial z^4} - \frac{E \rho I_p I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.28)$$

This equation represents the effect of shear deformation added to Timoshenko's torsion theory.

vi) The part of Eq.(4.23) given by:

$$-\frac{C_s \rho I_f}{K' A_f} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} + \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial z^4}$$

arises from the coupled interaction of torsional deformation with the bending effects of shear deformation and longitudinal inertia. The $\frac{\partial^4 \phi}{\partial t^4}$ term is responsible for introducing at high frequencies and short wave lengths, a new mode of wave transmission in long bars, and a completely new spectrum of natural frequencies in finite bars.

4.6. NON-DIMENSIONALIZATION AND GENERAL SOLUTION:

Eliminating ϕ in Eqs.(4.15) and (4.16) we obtain the complete differential equation in warping angle ψ as:

$$\left(\frac{EI_f C_s}{K' A_f} + EC_w\right) \frac{\partial^4 \psi}{\partial z^4} - \left(\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right) \frac{\partial^4 \psi}{\partial z^2 \partial t^2}$$

$$- GC_s \frac{\partial^2 \psi}{\partial z^2} + \rho I_p \frac{\partial^2 \psi}{\partial t^2} + \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4 \psi}{\partial t^4} = 0 \qquad (4.29)$$

Let

$$\phi = \bar{\phi} e^{ip_n t} \qquad (4.30)$$

$$\psi = \bar{\psi} e^{ip_n t} \qquad (4.31)$$

$$Z = z/L \qquad (4.32)$$

where $\bar{\phi}$ is the normal function of ϕ, $\bar{\psi}$ the normal function of ψ, Z the non-dimensional length of beam, $i = \sqrt{-1}$, p_n the natural frequency of vibration.

Substituting Eqs.(4.30) to (4.32) and omitting the factor $e^{ip_n t}$, Eqs.(4.15), (4.16), (4.23) and (4.29) are reduced to:

$$(s^2K^2+1)\bar{\phi}'' + \lambda^2 s^2\bar{\phi} - (2L/h)\bar{\psi}' = 0 \qquad (4.33)$$

$$s^2\bar{\psi}'' - (1-\lambda^2 s^2 d^2)\bar{\psi} + (h/2L)\bar{\phi}' = 0 \qquad (4.34)$$

$$(s^2K^2+1)\bar{\phi}^{iv} + \lambda^2(a^2d^2+s^2)\bar{\phi}'' - \lambda^2(1-\lambda^2 s^2d^2)\bar{\phi} = 0 \qquad (4.35)$$

$$(s^2K^2+1)\bar{\psi}^{iv} + \lambda^2(a^2d^2+s^2)\bar{\psi}'' - \lambda^2(1-\lambda^2 s^2d^2)\bar{\psi} = 0 \qquad (4.36)$$

where

$$a^2 = 1 + s^2K^2 - K^2/\lambda^2 d^2, \qquad (4.37)$$

$$\lambda^2 = \frac{\rho I_p L^4 p_n^2}{EC_w}, \text{ frequency parameter,} \qquad (4.38)$$

$$K^2 = \frac{L^2 GC_s}{EC_w}, \text{ warping parameter,} \qquad (4.39)$$

$$d^2 = \frac{I_f h^2}{2I_p L^2}, \text{ longitudinal inertia parameter,} \qquad (4.40)$$

$$s^2 = \frac{EI_f}{K' A_f G L^2}, \text{ shear deformation parameter} \qquad (4.41)$$

and the primes for $\bar{\phi}$ and $\bar{\psi}$ represent differentiation with respect to Z.

The general solutions of Eqs.(4.35) and (4.36) can be found as:

$$\bar{\phi} = A_1 \cosh \lambda\alpha_2 Z + A_2 \sinh \lambda\alpha_2 Z + A_3 \cos\lambda\beta_2 Z + A_4 \sin\lambda\beta_2 Z \qquad (4.42)$$

$$\bar{\psi} = A_1' \sinh\lambda\alpha_2 Z + A_2' \cosh\lambda\alpha_2 Z + A_3' \sin\lambda\beta_2 Z + A_4' \cos\lambda\beta_2 Z \qquad (4.43)$$

where

$$\begin{matrix}\alpha_2\\ \beta_2\end{matrix} = \frac{1}{\sqrt{2}(s^2K^2+1)^{1/2}}\left\{\mp(a^2d^2+s^2)+\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2}\right\}^{1/2} \tag{4.44}$$

and

$$\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2} > (a^2d^2+s^2)$$

is assumed.

In case

$$\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2} < (a^2d^2+s^2)$$

we write

$$\begin{aligned}\alpha_2 &= \frac{1}{\sqrt{2}(s^2K^2+1)^{1/2}}\left\{(a^2d^2+s^2)-\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2}\right\}^{1/2}\\ &= i\,\alpha_2'\end{aligned} \tag{4.45}$$

Then Eqs.(4.42) and (4.43) are replaced by

$$\bar{\phi} = A_1\cos\lambda\alpha_2'Z + i\,A_2\sin\lambda\alpha_2'Z + A_3\cos\lambda\beta_2Z + A_4\sin\lambda\beta_2Z \tag{4.46}$$

$$\bar{\psi} = i\,A_1'\sin\lambda\alpha_2'Z + A_2'\cos\lambda\alpha_2'Z + A_3'\sin\lambda\beta_2Z + A_4'\cos\lambda\beta_2Z \tag{4.47}$$

Solutions of Eqs.(4.42) and (4.43) or (4.46) and (4.47) are naturally the solutions of the original coupled equations (4.15) and (4.16).

Only one half of the constants in Eqs.(4.42) and (4.43) are independent. They are related by Eqs.(4.15) or (4.16) as follows:

$$A_1 = \frac{2L}{h\lambda\alpha_2}\left[1 - \lambda^2 s^2(\alpha_2^2 + d^2)\right] A_1' \tag{4.48}$$

$$A_2 = \frac{2L}{h\lambda\alpha_2}\left[1 - \lambda^2 s^2(\alpha_2^2 + d^2)\right] A_2' \tag{4.49}$$

$$A_3 = \frac{2L}{h\lambda\beta_2}\left[1 + \lambda^2 s^2(\beta_2^2 - d^2)\right] A_3' \tag{4.50}$$

$$A_4 = \frac{2L}{h\lambda\beta_2}\left[1 + \lambda^2 s^2(\beta_2^2 - d^2)\right] A_4' \tag{4.51}$$

or

$$A_1' = \frac{h\lambda}{2L}\left[\frac{\alpha_2^2(s^2K^2 + 1) + s^2}{\alpha_2}\right] A_1 \tag{4.52}$$

$$A_2' = \frac{h\lambda}{2L}\left[\frac{\alpha_2^2(s^2K^2 + 1) + s^2}{\alpha_2}\right] A_2 \tag{4.53}$$

$$A_3' = -\frac{h\lambda}{2L}\left[\frac{\beta_2^2(s^2K^2 + 1) - s^2}{\beta_2}\right] A_3 \tag{4.54}$$

$$A_4' = \frac{h\lambda}{2L}\left[\frac{\beta_2^2(s^2K^2 + 1) - s^2}{\beta_2}\right] A_4 \tag{4.55}$$

4.7. FREQUENCY EQUATIONS AND MODAL FUNCTIONS:

In section 4.4(a), natural boundary conditions were discussed. By combining these conditions in pairs, many types of single-span beams can be analyzed. In terms of non-dimensional parameters, the boundary conditions can be written as:

1. Simple Support:

$$\bar{\phi} = 0, \ \bar{\psi}' = 0 \qquad (4.56)$$

2. Fixed Support:

$$\bar{\phi} = 0, \ \bar{\psi} = 0 \qquad (4.57)$$

3. Free End:

$$\bar{\psi}' = 0, \ (s^2K^2+1)\bar{\phi}' - (2L/h)\bar{\psi} = 0 \qquad (4.58)$$

The application of appropriate boundary conditions (4.56) to (4.58) and, relations of integration constants (4.48) to (4.55), to equations (4.42) and (4.43) yields for each type of beam a set of four constants A_1 to A_4 with or without primes. In order that the solutions other than zero may exist the determinant of the coefficients of A's must be equal to zero. This leads to the frequency equations in each case and the roots of these frequency equation, λ_i, i = 1,2,3,...n, give the eigen values of the problem. The corresponding modal functions, $\bar{\phi}_i$ and $\bar{\psi}_i$, can be obtained accordingly.

4.7.1. SIMPLY SUPPORTED BEAM:

The boundary conditions for a beam simply supported at both ends are:

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 0$$

and

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 1$$

For the boundary conditions at $Z = 0$, Eqs.(4.42) and (4.43) give:

$$A_1 + A_3 = 0,$$

$$\left[\alpha_2^2(s^2K^2+1) + s^2\right]A_1 - \left[\beta_2^2(s^2K^2+1) - s^2\right]A_3 = 0$$

Since the secular determinant, ie., $(s^2K^2+1)(\alpha_2^2+\beta_2^2) \neq 0$, therefore it follows that: $A_1 = A_3 = 0.$ (4.59)

For the second pair of conditions at $Z = 1$, Eqs.(4.42) and (4.43) give:

$$A_2 \sinh \lambda\alpha_2 + A_4 \sin \lambda\beta_2 = 0,$$

and

$$\left[\alpha_2^2(s^2K^2+1) + s^2\right]A_2 \sinh \lambda\alpha_2 - \left[\beta_2^2(s^2K^2+1) - s^2\right]A_4 \sin \lambda\beta_2 = 0. \quad .. \quad (4.60)$$

For a non-trivial solution, the secular determinant must vanish. This gives the characterestic equation:

$$(s^2K^2+1)(\alpha_2^2 + \beta_2^2) \sinh\lambda\alpha_2 \sin\lambda\beta_2 = 0 \qquad (4.61)$$

Since $(s^2K^2+1)(\alpha_2^2 + \beta_2^2) \neq 0$, the possible solutions are:

$\lambda\alpha_2 = 0, \quad \lambda\beta_2 = 0;$

$\lambda\alpha_2 = 0, \quad \lambda\beta_2 \neq 0;$

$\lambda\alpha_2 \neq 0, \quad \lambda\beta_2 = 0;$

$\lambda\alpha_2 \neq 0, \quad \lambda\beta_2 = n\pi, \; n=1,2,3,\ldots$

The solution $\lambda\alpha_2=0$, $\lambda\beta_2 = 0$ is not valid and the cases $\lambda\alpha_2 \neq 0$, $\lambda\beta_2 = 0$ and $\lambda\alpha_2 = 0$, $\lambda\beta_2 \neq 0$, by Eq.(4.44) imply $\lambda^2 = 0$ and

$\lambda^2 = 1/s^2d^2$ respectively. Using the Eqs.(4.42) and (4.43) and following the above procedure for $\lambda^2 = 0$, and for $\lambda^2 = 1/s^2d^2$, we can see that the former case leads to a trivial solution and the latter to:

$$\bar{\phi} = 0, \quad \bar{\psi} = \text{constant} \tag{4.62}$$

The critical frequency $\lambda_c^2 = 1/s^2d^2$ thus represents the first thickness shear mode of the flanges (100). The existence of this mode for the simply supported case of Timoshenko beam in flexural vibrations has been demonstrated by Trail-Nash and Collar (3). It is overlooked by Anderson (3) and neglected by Dolph (3) by a wrong interpretation of the associate results.

The last case:

$$\lambda\alpha_2 \neq 0, \quad \lambda\beta_2 = n\pi, \quad n=1,2,3,\ldots \tag{4.63}$$

leads to the main solution of the problem. Letting $\lambda^2\beta^2 = -n^2\pi^2$ in Eq.(4.44), the frequency equation in λ^2 is obtained as:

$$s^2d^2\lambda^4 - \lambda^2\left[1 + n^2\pi^2(s^2+d^2+s^2d^2K^2)\right] + n^2\pi^2\left[n^2\pi^2(s^2K^2+1) + K^2\right] = 0 \tag{4.64}$$

This equation gives two real positive roots:

$$\lambda_{mn}^2 = \frac{1}{2\,s^2d^2}\left[\left\{1 + n^2\pi^2(s^2+d^2+s^2d^2K^2)\right\} + (-1)^m\left\{\left[1 + n^2\pi^2(s^2-d^2-s^2d^2K^2)\right]^2 + 4n^2\pi^2d^2\right\}^{1/2}\right] \tag{4.65}$$

This frequency equation (4.65) in λ^2, has an infinite number of roots which in general represent two coupled frequency spectra. It may noted that the roots λ^2_{2n} is always $> 1/s^2d^2$. The roots greater than the critical value are also admissible since the same frequency equation is obtained for the case $\lambda^2 > 1/s^2d^2$. Thus, both the roots of (4.65) are admitted and constitute the two uncoupled frequency spectra.

Using (4.63) and (4.60) one gets:

$$A_2 = 0. \tag{4.66}$$

The modal functions are obtained from Eqs.(4.42) and (4.43) with A's given by (4.59) and (4.66). These are given as:

$$\bar{\phi}_{mn} = \sin n\pi Z \tag{4.67}$$

$$\bar{\psi}_{mn} = \frac{h}{2n\pi L}\left[n^2\pi^2(s^2K^2+1) - \lambda^2_{mn}\, s^2\right]\cos n\pi Z \tag{4.68}$$

where λ^2_{mn} being given by (4.65).

The second spectrum appears at higher frequencies, greater than the critical frequency λ_c given by

$$\lambda^2_c = 1/s^2d^2 \tag{4.69}$$

and is due to interaction between shear deformation and longitudinal inertia. Eq.(4.69) therefore shows the thickness shear nature of the critical frequency while Eq.(4.65) shows the two frequency spectra, uncoupled in the present case.

The classical Timoshenko torsion theory provides only one set of frequency spectrum, while the present analysis provides two frequency spectra. The eigen values λ of the first set of frequency spectrum cover the whole range from zero to infinity, but those of the second set range from the critical frequency λ_c given by equation (4.69) to infinity.

For this case of a simply supported beam, Aggarwal (3), Tso (104) and Krishna Murty and Joga Rao (70) also illustrated two sets of frequency spectra. It is to be mentioned here that for the range of the values of the dimensionless parameters covered in this Chapter, λ is less than λ_c.

For the case, $\lambda > \lambda_c$, it is convenient to use $\alpha_2 = i\alpha_2'$ and, the characterestic frequency equation (4.61) transforms to:

$$\sin \lambda \alpha_2' \sin \lambda \beta_2 = 0 \qquad (4.70)$$

where α_2' is given by Eq.(4.45).

Hence, in case there is any extension from there on for λ beyond λ_c ie., $\lambda^2 s^2 d^2 > 1$, care should be taken to account for the frequencies of the second spectrum which can be obtained from Eq.(4.70).

By putting $s^2 = d^2 = 0$ in Eq.(4.64), the equation for the frequency parameter λ, neglecting the effects of shear deformation and longitudinal inertia, can be obtained as:

$$\lambda^2 = n^2\pi^2 \ (n^2\pi^2 + K^2) \qquad (4.71)$$

which is the same as that derived by Gere (32) utilizing Timoshenko torsion theory.

4.7.2. FIXED-FIXED BEAM:

In the case of a beam which is built-in rigidly at both ends, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0,$$

and

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 1.$$

Applying the above boundary conditions to the general solutions, Eqs.(4.42) and (4.43), the frequency equation, for the first set $(\lambda < \lambda_c)$, can be obtained as:

$$2 - 2 \cosh \lambda\alpha_2 \cos \lambda \beta_2$$
$$+ \frac{\lambda\left[\lambda^2 s^2(s^2-a^2d^2)+(3s^2-a^2d^2)\right]}{(1-\lambda^2s^2d^2)^{1/2}(s^2K^2+1)^{1/2}} \sinh \lambda\alpha_2 \sin \lambda\beta_2 = 0 \qquad (4.72)$$

The frequency equation for the second set $(\lambda > \lambda_c)$ is:

$$2 - 2 \cos \lambda\alpha_2' \cos \lambda \beta_2$$
$$+ \frac{\left[\lambda^2 s^2(s^2-a^2d^2)+(3s^2-a^2d^2)\right]}{(\lambda^2s^2d^2-1)^{1/2}\,(s^2K^2+1)^{1/2}} \sin \lambda\alpha_2' \sin \lambda \beta_2 = 0 \qquad (4.73)$$

The modal functions for the first set are given by:

$$\bar{\phi} = B(\cosh \lambda\alpha_2 Z + \delta\eta_1\theta \sinh \lambda\alpha_2 Z - \cos \lambda\beta_2 Z + \eta_1 \sin \lambda\beta_2 Z) \qquad (4.74)$$

$$\bar{\psi} = C(\cosh \lambda\alpha_2 Z + \frac{\mu_1}{\delta\theta} \sinh \lambda\alpha_2 Z - \cos \lambda\beta_2 Z + \overset{\mu_1}{\cancel{\eta_1}} \sin \lambda\beta_2 Z) \qquad (4.75)$$

where

$$\delta = \alpha_2/\beta_2$$

$$\theta = \frac{\beta_2^2(s^2K^2+1)-s^2}{\alpha_2^2(s^2K^2+1)+s^2} = \frac{\alpha_2^2(s^2K^2+1)+a^2d^2}{\beta_2^2(s^2K^2+1)-a^2d^2}$$

$$= \frac{\beta_2^2(s^2K^2+1)-s^2}{\beta_2^2(s^2K^2+1)-a^2d^2} = \frac{\alpha_2^2(s^2K^2+1)+a^2d^2}{\alpha_2^2(s^2K^2+1)+s^2}$$

$$\eta_1 = \frac{-\cosh\lambda\alpha_2+\cos\lambda\beta_2}{\delta\theta\sinh\lambda\alpha_2-\sin\lambda\beta_2}$$

$$\mu_1 = \frac{-\cosh\lambda\alpha_2+\cos\lambda\beta_2}{(1/\delta\theta)\sinh\lambda\alpha_2+\sin\lambda\beta_2}$$

The modal functions for the second set are:

$$\bar{\emptyset} = B(\cos\lambda\alpha_2'Z-\delta'\eta_2\theta\sin\lambda\alpha_2'Z-\cos\lambda\beta_2Z+\eta_2\sin\lambda\beta_2Z)$$

$$\bar{\ } = C(\cos\lambda\alpha_2'Z+\frac{\mu_2}{\delta'\theta}\sin\lambda\alpha_2'Z-\cos\lambda\beta_2Z+\mu_2\sin\lambda\beta_2Z)$$

where

$$\delta' = \alpha_2'/\beta_2$$

$$\eta_2 = \frac{\cos\lambda\alpha_2'-\cos\lambda\beta_2}{\delta'\theta\sin\lambda\alpha_2'-\sin\lambda\beta_2}$$

$$\mu_2 = \frac{-\cos\lambda\alpha_2'+\cos\lambda\beta_2}{(1/\delta'\theta)\sin\lambda\alpha_2'+\sin\lambda\beta_2}$$

Since the coefficients in $\bar{\phi}$ and $\bar{\psi}$ of Eqs. (4.42) and (4.43) are related, the constants B and C, that appear in the modal functions given above are connected through any one of the equations of (4.48) to (4.51) or (4.52) to (4.55).

4.7.3. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

With the end Z = 0, taken as built-in end, and the end Z = 1 as the simply supported end, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at} \quad Z = 1.$$

The frequency equation obtained from applying the above boundary condition to the general solutions, Eqs. (4.42) and (4.43), for the first set ($\lambda < \lambda_c$) is given by:

$$\delta\theta \tanh \lambda\alpha_2 - \tan \lambda\beta_2 = 0 \tag{4.85}$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$\delta'\theta \tanh \lambda\alpha_2' + \tan \lambda\beta_2 = 0 \tag{4.86}$$

The modal functions for the first set are given by:

$$\bar{\phi} = B(\cosh \lambda\alpha_2 Z - \coth \lambda\alpha_2 \sinh \lambda\alpha_2 Z - \cos \lambda\beta_2 Z + \cot \lambda\beta_2 \sin \lambda\beta_2 Z) \tag{4.87}$$

$$\bar{\psi} = C(\cosh \lambda\alpha_2 Z + \frac{\mu_3}{\delta\theta} \sinh \lambda\alpha_2 Z - \cos \lambda\beta_2 Z + \mu_3 \sin \lambda\beta_2 Z) \tag{4.88}$$

where

$$\mu_3 = \frac{-(\delta \sinh \lambda\alpha_2 + \sin \lambda\beta_2)}{(1/\theta)\cosh \lambda\alpha_2 + \cos \lambda\beta_2} \qquad (4.89)$$

The modal functions for the second set are:

$$\bar{\phi} = B(\cos \lambda\alpha_2' Z - \cot \lambda\alpha_2' \sin \lambda\alpha_2' Z - \cos \lambda\beta_2 Z$$

$$+ \cot \lambda\beta_2 \sin \lambda\beta_2 Z) \qquad (4.90$$

$$\bar{\psi} = C(\cos \lambda\alpha_2' Z - \frac{\eta_3}{\delta'\theta} \sin \lambda\alpha_2' Z - \cos \lambda\beta_2 Z + \eta_3 \sin \lambda\beta_2 Z) \qquad (4.91$$

where

$$\eta_3 = \frac{\delta' \sin \lambda\alpha_2' - \sin \lambda\beta_2}{(1/\theta)\cos \lambda\alpha_2' + \cos \lambda\beta_2} \qquad (4.92$$

4.7.4. CANTILEWER BEAM WITH ONE END FIXED AND FREE AT THE OTHER:

For a Cantilever beam built-in rigidly at the end Z=0 so that warping is completely prevented, and with a free end at Z = 1, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0,$$

and

$$\bar{\ }' = 0, \ (s^2K^2 + 1)\,\bar{\phi}' - (2L/h)\,\bar{\psi} = 0 \quad \text{at} \quad Z = 1.$$

The frequency equation for the first set, in this case, can be obtained as:

$$2 + \left[\lambda^2(a^2d^2 - s^2) + 2\right] \cosh \lambda\alpha_2 \cos \lambda\beta_2$$

$$- \frac{(a^2d^2 + s^2)\,\lambda}{(1-\lambda^2 s^2 d^2)^{1/2}\,(s^2K^2 + 1)^{1/2}} \sinh \lambda\alpha_2 \sin \lambda\beta_2 = 0 \quad (4.$$

The frequency equation for the second set is given by:

$$2 + \left[\lambda^2(a^2d^2 - s^2) + 2\right]\cos\lambda\alpha_2' \cos\lambda\beta_2 - \frac{\lambda(a^2d^2 + s^2)}{(\lambda^2 s^2 d^2 - 1)^{1/2}(s^2K^2+1)^{1/2}} \sin\lambda\alpha_2' \sin\lambda\beta_2 = 0 \qquad (4.94)$$

The modal functions for the first set are:

$$\bar{\phi} = B(\cosh\lambda\alpha_2 Z - \delta\theta\,\eta_4 \sinh\lambda\alpha_2 Z - \cos\lambda\beta_2 Z + \eta_4 \sin\lambda\beta_2 Z) \qquad (4.95)$$

$$\bar{\psi} = C(\cosh\lambda\alpha_2 Z + \frac{\mu_4}{\delta\theta}\sinh\lambda\alpha_2 Z - \cos\lambda\beta_2 Z + \mu_4 \sin\lambda\beta_2 Z) \qquad (4.96)$$

where

$$\eta_4 = \frac{(1/\delta)\sinh\lambda\alpha_2 - \sin\lambda\beta_2}{\theta\cosh\lambda\alpha_2 + \cos\lambda\beta_2} \qquad (4.97)$$

$$\mu_4 = -\frac{(\delta\sinh\lambda\alpha_2 + \sin\lambda\beta_2)}{(1/\theta)\cosh\lambda\alpha_2 + \cos\lambda\beta_2} \qquad (4.98)$$

The modal functions for the second set are:

$$\bar{\phi} = B(\cos\lambda\alpha_2' Z + \delta'\theta\,\eta_5 \sin\lambda\alpha_2' Z - \cos\lambda\beta_2 Z + \eta_5 \sin\lambda\beta_2 Z) \qquad (4.99)$$

$$\bar{\psi} = C(\cos\lambda\alpha_2' Z - \frac{\mu_5}{\delta'\theta}\sin\lambda\alpha_2' Z - \cos\lambda\beta_2 Z + \mu_5 \sin\lambda\beta_2 Z) \qquad (4.100)$$

where

$$\eta_5 = \frac{(1/\delta')\sin\lambda\alpha_2' - \sin\lambda\beta_2}{\theta\cos\lambda\alpha_2' + \cos\lambda\beta_2} \qquad (4.101)$$

$$\mu_5 = \frac{\delta'\sin\lambda\alpha_2' - \sin\lambda\beta_2}{(1/\theta)\cos\lambda\alpha_2' + \cos\lambda\beta_2} \qquad (4.102)$$

4.7.5. CANTILEVER BEAM WITH ONE END SIMPLY SUPPORTED AND FREE AT THE OTHER:

For a Cantilever beam simply supported at the end Z=0 and free at Z=1, the boundary conditions are:

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 0,$$

and

$$\bar{\psi}' = 0, \; (s^2K^2 + 1)\bar{\phi}' - (2L/h)\bar{\psi} \quad \text{at } Z = 1.$$

The frequency equation for the first set, in this case becomes:

$$\delta \tanh \lambda\alpha_2 - \theta \tan \lambda\beta_2 = 0 \tag{4.103}$$

The frequency equation for the second set is given by:

$$\delta' \tan \lambda\alpha_2' + \theta \tan \lambda\beta_2 = 0 \tag{4.104}$$

The modal functions for the first set are:

$$\bar{\phi} = \frac{\delta \cos\lambda\beta_2}{\cosh\lambda\alpha_2} \sinh\lambda\alpha_2 Z + \sin\lambda\beta_2 Z \tag{4.105}$$

$$\bar{\psi} = \frac{\sin\lambda\beta_2}{\delta \sinh\lambda\alpha_2} \cosh\lambda\alpha_2 Z + \cos\lambda\beta_2 Z \tag{4.106}$$

The modal functions for the second set can be obtained a

$$\bar{\phi} = -\frac{\delta' \cos\lambda\beta_2}{\cos\lambda\alpha_2'} \sin\lambda\alpha_2' Z + \sin\lambda\beta_2 Z \tag{4.107}$$

$$\bar{\psi} = -\frac{\sin\lambda\beta_2}{\delta' \sin\lambda\alpha_2'} \cos\lambda\alpha_2' Z + \cos\lambda\beta_2 Z \tag{4.108}$$

4.7.6. BEAM WITH FREE ENDS:

In the case of a beam which is free at both ends, the boundary conditions are:

$$\bar{\psi}' = 0, \quad (s^2K^2+1)\bar{\phi}' - (2L/h)\,\bar{\psi} = 0 \quad \text{at } Z = 0,$$

and

$$\bar{\psi}' = 0, \quad (s^2K^2+1)\bar{\phi}' - (2L/h)\bar{\psi} = 0 \quad \text{at } Z = 1.$$

The frequency equation for the first set, in this case can be obtained as:

$$2 - 2\cosh\lambda\alpha_2\cos\lambda\beta_2 + \frac{\lambda\left[\lambda^2a^2d^2(a^2d^2-s^2)+(3a^2d^2-s^2)\right]}{(1-\lambda^2s^2d^2)^{1/2}\,(s^2K^2+1)^{1/2}}\sinh\lambda\alpha_2\sin\lambda\beta_2 = 0 \tag{4.109}$$

The frequency equation for the second set is given by:

$$2 - 2\cos\lambda\alpha_2'\cos\lambda\beta_2 + \frac{\lambda\left[\lambda^2a^2d^2(a^2d^2-s^2)^2+(3a^2d^2-s^2)\right]}{(\lambda^2s^2d^2-1)^{1/2}\,(s^2K^2+1)^{1/2}}\sin\lambda\alpha_2\sin\lambda\beta_2 = 0 \tag{4.110}$$

The modal functions for the first set can be obtained as:

$$\bar{\phi} = B\left(\cosh\lambda\alpha_2Z - \frac{\eta_6}{\delta}\sinh\lambda\alpha_2Z + \frac{1}{\delta}\cos\lambda\beta_2Z + (1/\eta_6)\sin\lambda\beta_2Z\right) \tag{4.111}$$

$$\bar{\psi} = C\left(\cosh\lambda\alpha_2Z - \frac{\eta_6}{\delta}\sinh\lambda\alpha_2Z + 0\cos\lambda\beta_2Z + (1/\eta_6)\sin\lambda\beta_2Z\right) \tag{4.112}$$

where

$$\eta_6 = \frac{\cosh \lambda \alpha_2 - \cos \lambda \beta_2}{\delta \sinh \lambda \alpha_2 - \theta \sin \lambda \beta_2} \qquad (4.113)$$

The modal functions for the second set are given by:

$$\bar{\phi} = B(\cos \lambda \alpha_2' Z - \delta' \mu_6 \sin \lambda \alpha_2' Z + (1/\theta) \cos \lambda \beta_2 Z + \mu_6 \sin \lambda \beta_2 Z) \qquad (4.114)$$

$$\bar{\psi} = C(\cos \lambda \alpha_2' Z - (\mu_6/\delta') \sin \lambda \alpha_2' Z + \theta \cos \lambda \beta_2 Z + (1/\mu_6) \sin \lambda \beta_2 Z) \qquad (4.11$$

where

$$\mu_6 = \frac{\cos \lambda \alpha_2' - \cos \lambda \beta_2}{\delta' \sin \lambda \alpha_2' + \theta \sin \lambda \beta_2} \qquad (4.116)$$

4.8. ORTHOGONALITY AND NORMALIZING CONDITIONS*:

In this section, the expressions for orthogonality and normalizing conditions for the principal normal modes $\bar{\phi}$ and $\bar{\psi}$ are obtained for both the general case and for beams with various simple end conditions.

Let Eq.(4.33) be written in the form

$$\lambda^2 s^2 \bar{\phi} = (2L/h) \bar{\psi}' - (s^2 K^2 + 1) \bar{\phi}''$$

for two modes m and n as,

$$\lambda_m^2 s^2 \bar{\phi}_m = (2L/h) \bar{\psi}_m' - (s^2 K^2 + 1) \bar{\phi}_m'' \qquad (4.117)$$

$$\lambda_n^2 s^2 \bar{\phi}_m = (2L/h) \bar{\psi}_n' - (s^2 K^2 + 1) \bar{\phi}_n'' \qquad (4.118)$$

---

* Results from this part of the Chapter were presented by the author and K.V.Apparao at the 16th Congress of ISTAM held at M.N.R.Engineering College, Allahabad, during 29th March to 1st April, 1972. See Ref.(50).

Multiplying Eq.(4.117) by $\bar{\phi}_n$ and Eq.(4.118) by $\bar{\phi}_m$ and subtracting Eq.(4.117) from Eq.(4.118), we have:

$$(\lambda_n^2-\lambda_m^2)\, s^2\bar{\phi}_m\bar{\phi}_n = (2L/h)(\bar{\psi}_n'\,\bar{\phi}_m - \bar{\psi}_m'\,\bar{\phi}_n) - (s^2K^2+1)(\bar{\phi}_n''\,\bar{\phi}_m - \bar{\phi}_m''\,\bar{\phi}_n) \tag{4.119}$$

Let Eq.(4.34) be written in the form

$$\lambda^2 s^2 d^2 \bar{\psi} = \bar{\psi} - s^2\bar{\psi}'' - (h/2L)\,\bar{\phi}'$$

for the two modes m and n as,

$$\lambda_m^2\, s^2 d^2\, \bar{\psi}_m = \bar{\psi}_m - s^2\bar{\psi}_m'' - (h/2L)\,\bar{\phi}_m' \tag{4.120}$$

$$\lambda_n^2\, s^2 d^2\, \bar{\psi}_n = \bar{\psi}_n - s^2\,\bar{\psi}_n'' - (h/2L)\bar{\phi}_n' \tag{4.121}$$

Multiplying Eq.(4.120) by $\bar{\psi}_n$ and Eq.(4.121) by $\bar{\psi}_m$ and subtracting Eq.(4.120) from (4.121), we get:

$$\begin{aligned}(\lambda_n^2-\lambda_m^2)\, s^2\Omega^2\, \bar{\psi}_m\,\bar{\psi}_n &= (2L/h)(\bar{\phi}_m'\,\bar{\psi}_n - \bar{\phi}_n'\,\bar{\psi}_m)\\ &\quad - (4s^2L^2/h^2)(\bar{\psi}_n''\,\bar{\psi}_m - \bar{\psi}_m''\,\bar{\psi}_n)\end{aligned} \tag{4.122}$$

where

$$\Omega^2 = (4L^2/h^2)d^2 = 2I_f/I_p \tag{4.123}$$

Combining Eqs.(4.119) and (4.122), integrating over the whole beam, and carrying out integration by parts for most of the terms, we obtain:

$$(\lambda^2_n - \lambda^2_m)s^2 \int_0^1 (\bar{\phi}_m \bar{\phi}_n + \Omega^2 \bar{\psi}_m \bar{\psi}_n)\, dZ$$

$$= \int_0^1 \Big[(2L/h)(\bar{\psi}'_n \bar{\phi}_m + \bar{\psi}_n \bar{\phi}'_m) - (2L/h)(\bar{\psi}'_m \bar{\phi}_n + \bar{\psi}_m \bar{\phi}'_n)$$

$$- (s^2K^2+1)(\bar{\phi}''_n \bar{\phi}_m - \bar{\phi}_n \bar{\phi}''_m) - (4s^2L^2/h^2)(\bar{\psi}''_n \bar{\psi}_m - \bar{\psi}_n \bar{\psi}''_m)\Big]\, dZ$$

$$= \Big[(2L/h)(\bar{\psi}_n \bar{\phi}_m - \bar{\phi}_n \bar{\psi}_m) - (s^2K^2+1)(\bar{\phi}'_n \bar{\phi}_m - \phi_n \bar{\phi}'_m)$$

$$- (4s^2L^2/h^2)(\bar{\psi}'_n \bar{\psi}_m - \bar{\psi}_n \bar{\psi}'_m)\Big]\Big|_0^1 \qquad (4.124)$$

Applying end conditions of any combinations gives the orthogonality condition:

$$\int_0^1 (\bar{\phi}_m \bar{\phi}_n + \Omega^2 \bar{\psi}_m \bar{\psi}_n)\, dZ = 0,\ m \neq n \qquad (4.125)$$

For m = n, the left side of the equations is identically equal to zero because $\lambda_m = \lambda_n$.

Thus the normalizing integral:

$$\int_0^1 (\phi^2 + \Omega^2 \bar{\psi}^2)\, dZ$$

cannot be obtained directly by putting m = n in Eq.(4.125)

To evaluate this integral, we let

$$\lambda_m = \lambda \qquad (4.126)$$

$$\lambda_n = \lambda + \bar{\delta}\lambda \qquad (4.127)$$

in which $\bar{\delta}\lambda$ is a small variation of λ, and $\lambda_n = \lambda_m$ as $\bar{\delta}\lambda$ approaches zero. Thus, we have

$$\lambda^2_m = \lambda^2 \tag{4.128}$$

$$\lambda^2_n = (\lambda + \bar{\delta}\lambda)^2 = \lambda^2 + 2\lambda\bar{\delta}\lambda \tag{4.129}$$

in which the higher order small term in the expression of $^{2}_{n}$ is omitted. We also have:

$$\bar{\phi}_n = \bar{\phi}_m + \frac{d\bar{\phi}_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.130}$$

$$\bar{\psi}_n = \bar{\psi}_m + \frac{d\bar{\psi}_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.131}$$

$$\bar{\phi}'_n = \bar{\phi}'_m + \frac{d\bar{\phi}'_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.132}$$

$$\bar{\psi}_n = \bar{\psi}_m + \frac{d\bar{\psi}'_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.133}$$

where

$$\frac{d}{d\lambda} = \frac{\partial}{\partial\lambda} + \frac{d\alpha_2}{d\lambda} \cdot \frac{\partial}{\partial\alpha_2} + \frac{d\beta_2}{d\lambda} \cdot \frac{\partial}{\partial\beta_2} \tag{4.134}$$

Substituting the above relations in Eq.(4.124) we obtain:

$$\begin{aligned} & 2\lambda\bar{\delta}\lambda s^2 \int_0^1 (\bar{\phi}^2_m + \Omega^2 \bar{\psi}^2_m)\, dZ \\ & = \Bigg[(2L/h)\Big(\frac{d\bar{\psi}_m}{d\lambda}\bar{\phi}_m - \frac{d\bar{\phi}_m}{d\lambda}\bar{\psi}_m\Big) - (s^2K^2+1)\Big(\frac{d\bar{\phi}'_m}{d\lambda}\bar{\phi}_m - \frac{d\bar{\phi}_m}{d\lambda}\bar{\phi}'_m\Big) \\ & \qquad - (4s^2L^2/h^2)\Big(\frac{d\bar{\psi}'_m}{d\lambda}\bar{\psi}_m - \frac{d\bar{\psi}_m}{d\lambda}\bar{\psi}'_m\Big) \Bigg] \Bigg|_0^1 \delta\lambda \end{aligned} \tag{4.135}$$

Dropping the subscript m, dividing both sides of the equation by

$2\lambda\delta\lambda s^{2}$, and rearranging:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda s^2}\left\{\bar{\phi}\frac{d}{d\lambda}\left[(2L/h)\bar{\psi} - (s^2K^2+1)\bar{\phi}'\right]\right.$$

$$\left. + \left[(s^2K^2+1)\bar{\phi}' - (\frac{2L}{h})\bar{\psi}\right]\frac{d\bar{\phi}}{d\lambda} - (\frac{4s^2L^2}{h^2})\left[\frac{d\bar{\psi}'}{d\lambda}\bar{\psi} - \frac{d\bar{\psi}}{d\lambda}\bar{\psi}'\right]\right\}\Bigg|_0^1 \delta \qquad (4.136)$$

This expression can be further simplified for beams of various end conditions as follows:

(1) Simply Supported beam:

$$\int_0^1(\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda^2 s^2}\left\{\left[(s^2K^2+1)\bar{\phi}' - (\frac{2L}{h})\bar{\psi}\right]\frac{d\bar{\phi}}{d\lambda}\right.$$

$$\left. + (\frac{4s^2L^2}{h^2})\bar{\psi}\frac{d\bar{\psi}'}{d\lambda}\right\}\Bigg|_0^1 \qquad (4.137)$$

(2) **Fixed-End Beam:** :

$$\int_0^1(\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda^2 s^2}\left[(s^2K^2+1)\bar{\phi}'\frac{d\bar{\phi}'}{d\lambda} +\right.$$

$$\left. + (\frac{4s^2L^2}{h^2})\,\bar{}'\,\frac{d\bar{\psi}}{d\lambda}\right]\Bigg|_0^1 \qquad (4.138)$$

(3) Beam Free at both ends:

$$\int_0^1(\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda^2 s^2}\left[\bar{\phi}\frac{d}{d\lambda}\left|(\frac{2L}{h})\bar{\psi} - (s^2K^2+1)\bar{\phi}'\right]\right.$$

$$- \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}\;\frac{d\bar{\psi}'}{d\lambda}\Bigg]_0^1 \tag{4.139}$$

(4) Beam fixed at one end, simply supported at the other:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2\bar{\psi}^2)\,dZ = \frac{1}{2\lambda^2 s^2}\left[\left\{\left[(s^2K^2+1)\bar{\phi}' - \left(\frac{2L}{h}\right)\bar{\psi}\right]\frac{d\bar{\phi}}{d\lambda} + \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}\;\frac{d\bar{\psi}'}{d\lambda}\right\}_{z=1} - \left\{(s^2K^2+1)\bar{\phi}'\;\frac{d\bar{\phi}}{d\lambda} + \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}'\;\frac{d\bar{\psi}}{d\lambda}\right\}_{Z=0}\right] \tag{4.140}$$

(5) Cantilever beam fixed at one end, free at the other:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2\bar{\psi}^2)\,dZ = \frac{1}{2\lambda s^2}\left[\left\{\bar{\phi}\;\frac{d}{d\lambda}\left|\left(\frac{2L}{h}\right)\bar{\psi} - (s^2K^2+1)\,\bar{\phi}'\right] - \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}\;\frac{d\bar{\psi}}{d\lambda}\right]_{Z=1} - \left\{(s^2K^2+1)\bar{\phi}'\;\frac{d\bar{\phi}}{d\lambda} + \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}'\;\frac{d\bar{\psi}}{d\lambda}\right\}_{Z=0}\right] \tag{4.141}$$

(6) Cantilever beam simply supported at one end, free at the other:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2\bar{\psi}^2)\,dZ = \frac{1}{2\lambda^2 s^2}\left[\bar{\phi}\;\frac{d}{d\lambda}\left[\left(\frac{2L}{h}\right)\bar{\psi} - (s^2K^2+1)\,\bar{\phi}'\right] - \left(\frac{4s^2L^2}{h^2}\right) - \frac{d\bar{\psi}'}{d\lambda}\right\}_{Z=1} - \left\{\left[(s^2K^2+1)\bar{\phi}' - \left(\frac{2L}{h}\right)\bar{\psi}\right]\frac{d\bar{\phi}}{d\lambda} + \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}\;\frac{d\bar{\psi}'}{d\lambda}\right\}_{Z=0}\right] \tag{4.142}$$

It is also suggested that the normalizing integral can be approximated by discrete values of $\bar{\phi}$ and $\bar{\psi}$ along the beam.

Expression of Normalizing condition:

Let Eqs.(4.33) and (4.34) be written as:

$$\lambda^2 s^2 \bar{\phi} = -(s^2K^2+1)\bar{\phi}'' + (2L/h)\bar{\psi}' \qquad (4.143)$$

$$\lambda^2 s^2 d^2 \bar{\psi} = -s^2\bar{\psi}'' + \bar{\psi} - (h/2L)\bar{\phi}' \qquad (4.144)$$

Multiplying the Eq.(4.143) by $\bar{\phi}$ and the Eq.(4.144) by $\bar{\psi}$, adding the resulting equations, integrating over the whole beam, and carrying out some integrals by integration by parts, we have:

$$\lambda^2 s^2 \int_0^1 (\bar{\phi}^2 + \Omega^2\bar{\psi}^2)dZ = \int_0^1 \Big[-(s^2K^2+1)\bar{\phi}\bar{\phi}'' + (\tfrac{2L}{h})(\bar{\phi}\bar{\psi}' - \bar{\phi}'\bar{\psi}')$$

$$+ (\tfrac{4L^2}{h^2})\bar{\psi}^2 - (\tfrac{4s^2L^2}{h^2})\bar{\psi}\bar{\psi}'' \Big] dZ$$

$$= \int_0^1 \Big[(s^2K^2+1)\bar{\phi}'^2 - (\tfrac{4L}{h})\bar{\phi}'\bar{\psi} + (\tfrac{4s^2L^2}{h^2})\bar{\psi}'^2 + (\tfrac{4L^2}{h^2})\bar{\psi}^2 \Big] dZ \qquad (4.145)$$

Eq.(4.145) is the expression of the Normalizing condition which is very useful in analyzing the forced vibration problems.

4.9. APPROXIMATE SOLUTIONS BY GALERKIN'S TECHNIQUE*:

In this section, approximate solutions are obtained, for the problem of free torsional vibrations of thin-walled beams of open section including the effects of longitudinal inertia and shear deformation, utilizing the well-known Galerkin's technique. Solutions with Galerkin's method are illustrated for fixed-fixed beam and for a beam fixed at one end and simply supported at the other.

4.9.1. FIXED-FIXED BEAM:

To satisfy the above boundary conditions in this case, the normal function $\bar{\phi}$ can be assumed in the form

$$\bar{\phi} = \sum_{n=1}^{\infty} D_n (1 - \cos 2n\pi Z) \qquad (4.146)$$

Substituting Equation (4.146) in the differential Equation (4.35), orthogonalizing the resulting error with the assumed function, integrating the obtained function over the whole length of the beam and equating it to zero, the frequency equation in λ^2 can be obtained as:

$$3\lambda^4 s^2 d^2 - \lambda^2 \left| 3 + 4n^2\pi^2 (s^2 + d^2 + s^2 d^2 K^2) \right| + 4n^2\pi^2 \left| 4n^2\pi^2 (s^2 K^2 + 1) + K^2 \right| = 0 \qquad (4.147)$$

---

* Results from this part of the chapter were presented at the 17th Congress of Indian Society of Theoretical and Applied Mechanics, held at Birla Institute of Technology, Mesra, Ranchi, during December 22-25 1972. Ref (51)

Eq.(4.147) gives two real positive roots given by

$$\lambda^2_{mn} = \frac{1}{6s^2d^2}\left[\left\{3+4n^2\pi^2(s^2+d^2+s^2d^2K^2)\right\} + (-1)^m\left\{\left[3+4n^2\pi^2(s^2+d^2+s^2d^2K^2)\right]^2 - 48\,n^2\pi^2s^2d^2\left[4n^2\pi^2(s^2K^2+1)+K^2\right]\right\}^{1/2}\right] \qquad (4.148)$$

In arriving at Eq.(4.148), only one term of the infinite series of Eq.(4.146) is utilized. Hence, Eq.(4.148) gives upper bounds and has an infinite number of roots which in general represent two coupled frequency spectra.

By putting $s^2 = d^2 = 0$, Eq.(4.147) reduces to:

$$3\lambda^2 - 4\,n^2\pi^2(4n^2\pi^2 + K^2) = 0 \qquad (4.149)$$

and the expression for the frequency parameter λ becomes:

$$\lambda_n = \frac{2n\pi}{\sqrt{3}}\,(4n^2\pi^2+K^2)^{1/2} \qquad (4.150)$$

which is same as that from Eq.(2.73) for $\Delta^2 = \gamma^2 = 0$.

4.9.2. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

The normal function satisfying the boundary conditions in this case can be assumed in the form:

$$\bar{\phi} = \sum_{n=1}^{\infty} D_n\left(\cos\frac{n\pi}{2}Z - \cos\frac{3n\pi}{2}Z\right) \qquad (4.151)$$

Substituting Eq.(4.151) in the Eq.(4.35) and following the Galerkin's method, the frequency equation in λ^2 can be obtained as:

$$16\lambda^4 s^2 d^2 - \lambda^2 \left[16 + 20\, n^2\pi^2 (s^2 + d^2 + s^2 d^2 K^2) \right] + n^2\pi^2 \left[41\, n^2\pi^2 (s^2K^2 + 1) + 20\, K^2 \right] = 0 \qquad (4.152)$$

From Eq.(4.152) we have:

$$\lambda^2_{mn} = \frac{1}{16\, s^2 d^2} \Big[\Big\{ 16 + 20\, n^2\pi^2 (s^2 + d^2 + s^2 d^2 K^2) + (-1)^m \Big\{ \left[16 + 20\, n^2\pi^2 (s^2 + d^2 + s^2 d^2 K^2) \right]^2 - 64\, n^2\pi^2 s^2 d^2 \left[41\, n^2\pi^2 (s^2K^2 + 1) + 20\, K^2 \right] \Big\}^{1/2} \Big] \qquad (4.153)$$

By putting $s^2 = d^2 = 0$, Eq.(4.152) reduces to:

$$16\lambda^2 n^2\pi^2 (41\, n^2\pi^2 + 20\, K^2) = 0 \qquad (4.154)$$

and the expression for the frequency parameter λ becomes:

$$\lambda = \frac{n\pi}{4} (41\, n^2\pi^2 + 20\, K^2)^{1/2} \qquad (4.155)$$

which is same as that from Eq.(2.76) for $\Delta^2 = \gamma^2 = 0$.

4.10. RESULTS AND CONCLUSIONS:

For a given beam with K, s and d known, the $\lambda_i (i=1,2,3,\ldots)$ can be found from the appropriate frequency equations and the corresponding p_i are then calculated by Eq.(4.38). However, these frequency equations are highly transcendental and cannot ~~not to~~ be solved simply. This difficulty is overcome by the use of bisection method on digital Computer IBM 1130 at the Computer Center, Andhra University, Waltair. The results are obtained for some typical boundary conditions and various combinations of K, s and d. The results are presented for the special case s = 2d, which is usually the case for many Indian Standard wide-flanged I-beams.

Let λ_o be the classical eigen values obtained in Chapter II neglecting the effects of longitudinal inertia and shear deformation and p_o, the natural torsional frequencies corresponding to λ_o. Comparing the mechanism of vibration of the classical beam based on Timoshenko Torsion theory and the present beam based on the improved theory, we note that the classical beam is equivalent to the present beam with longitudinal inertia and shear constraints.

Therefore,

$$p \leqslant p_o$$

and

$$\lambda/\lambda_o = p/p_o = q, \; q < 1$$

The ratio of λ/λ_o or p/p_o, denoted by q, will be referred to the ''modifying quotient''. The variation of the ratio λ/λ_0 (also the modifying quotient q) with the longitudinal inertia parameter d for the first three modes of vibration of a simply supported beam is plotted in Fig.4.3, which shows the corrections in the natural torsional frequencies owing to the individual influence of longitudinal inertia. In plotting this figure the warping parameter is taken as equal to 1.0 and the shear parameter s as equal to zero. It can be observed from Fig.4.3 that the reduction in the torsional frequency due to longitudinal inertia increases with increasing values of d. For a maximum value of d = 0.1, the reduction in the torsional frequency can be observed from the graph as about 10 percent for the first mode, 35 percent for the second mode and 65 percent for the third mode. Therefore it can be concluded that the influence of longitudinal inertia on the torsional frequencies increases profoundly for higher modes of vibration.

For a simply supported beam, its higher harmonic corresponds to the fundamental of another simply supported beam of shorter span. The nth frequency of simply-supported beam of span L is equal to the fundamental of another such beam with span L/n. So, for the sake of simplicity and ease of presentation, Fig.4.4 is plotted between the ratio λ/λ_0 and K/n for values of ns = 0.5, 1.0 and 2.0. For constant values of K and s the values of λ/λ_0 can be read from this figure for different values of n (ie., for different modes of vibration). If n is kept constant, the values of λ/λ_0 can be obtained for various combinations of the warping parameter K and shear parameter s. In plotting

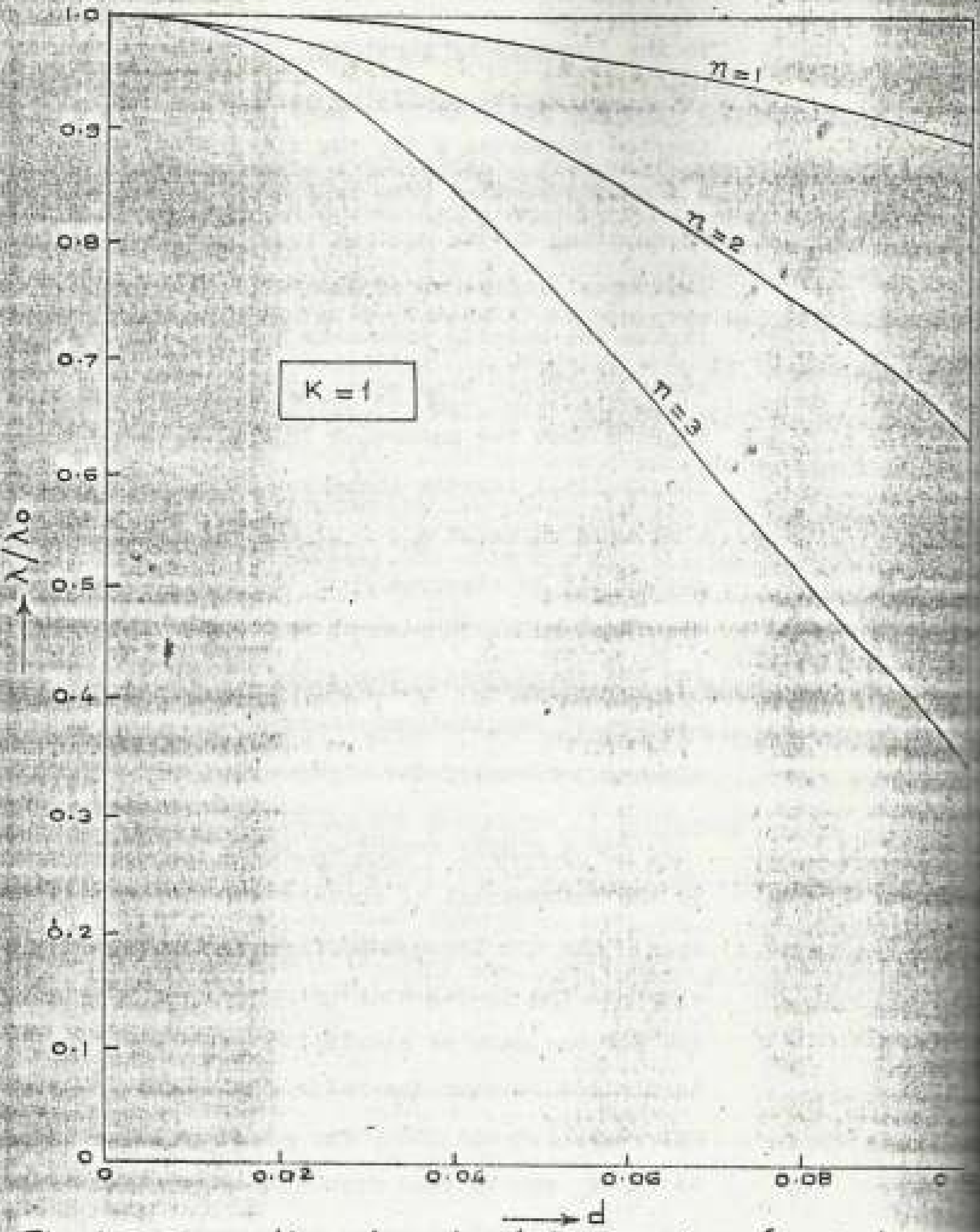


Fig. 4.3. Corrections in natural frequencies of a simply sup
beam owing to longitudinal inertia for the first three
of Vibration (S=0)

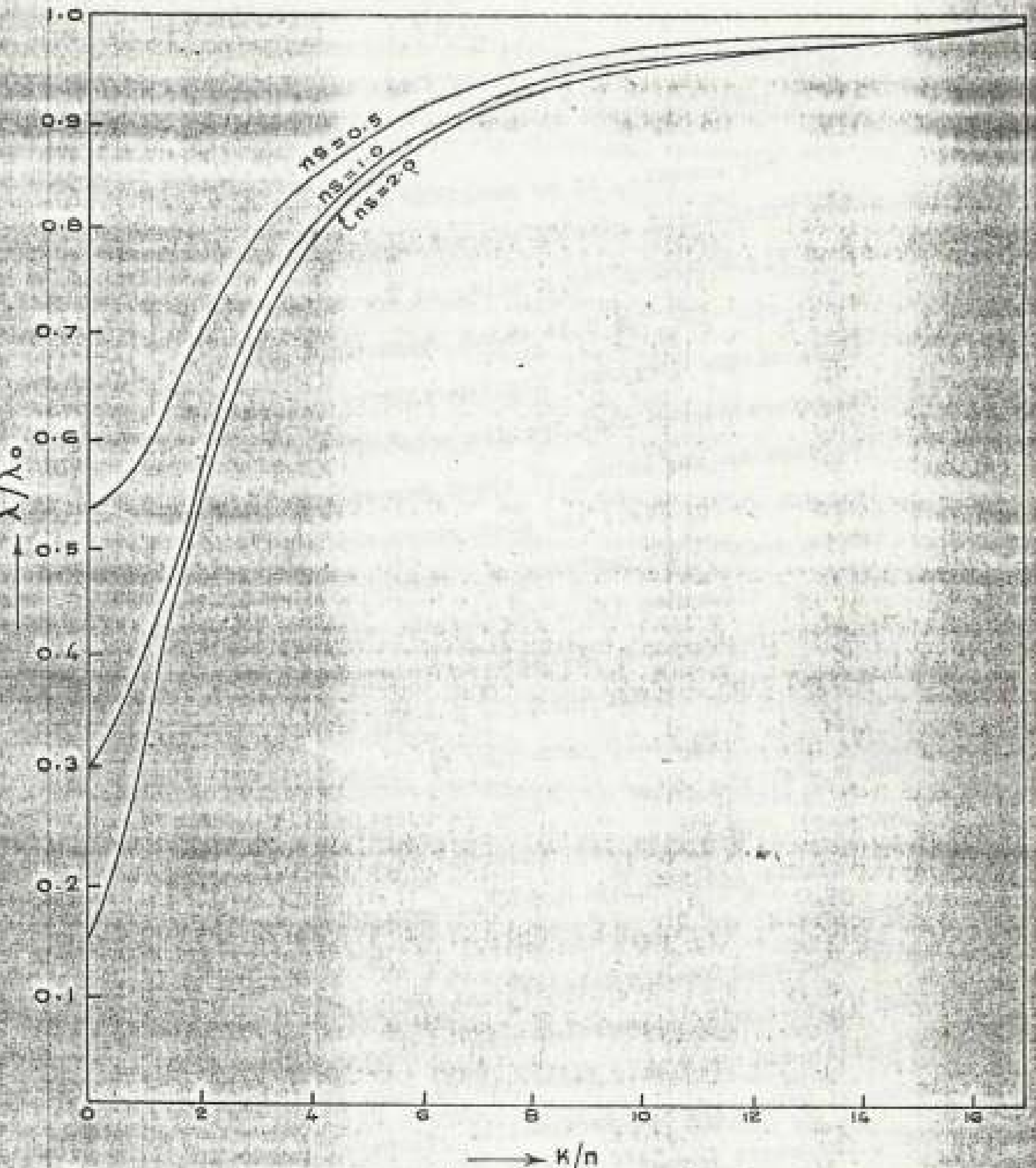


Fig. 4.4 Corrections in natural frequencies of a simply supported beam owing to shear deformation ($d=0$)

this graph, the value of the longitudinal inertia parameter d is taken as equal to zero.

For example, if we consider the variation of λ/λ_o for the fundamental mode of vibration (ie., n = 1), we can observe from Fig.4.4 that for a value of K = 1, and for s = 2.0, the value of λ/λ_o is 0.34 which means that the reduction in the value of the torsional frequency is by 66 percent. It can be therefore stated that for any constant value of n and K, the increase in the values of shear parameter s decreases the values of λ/λ_o (ie., the modifying quotient q). This reduction can be seen to be profound for smaller values of K and for higher modes of vibration (ie., for larger values of n). If the value of shear parameter s is taken as constant, say 0.5, it can be observed from Fig.4.4 that for K = 4.0 and n = 1, the value of λ/λ_o is 0.85 (ie., reduction is by 15 percent) and for K = 4.0, and n = 4, the value of λ/λ_o is 0.34 (ie., reduction is by 66 percent). It can be also observed that the increase in the value of mode number n and (or) decrease in the value of warping parameter K, decreases the values of λ/λ_o. It can be therefore concluded that the individual influence of shear deformation is to decrease the torsional frequency for any mode of vibration and that this reduction becomes significant for higher modes of vibration and for smaller values of warping parameter K (ie., for short beams). From Figs.4.3 and 4.4 we can observe that the effects of both longitudinal inertia and shear deformation is to decrease the frequency of vibration and that this reduction becomes significant for higher modes of vibration. It can be also observed that comparatively the individual influence of shear deformation on the torsional frequency of vibration is more profound than that of longitudinal inertia.

The combined effects of longitudinal inertia and shear deformation on the first four torsional frequencies of the first set of simply-supported, clamped-simply supported and clamped-clamped beams (s = 2d) are shown in Tables 4.1, 4.2 and 4.3 respectively. The values of the frequency parameter λ^2 and modified quotients $q = \lambda/\lambda_o$ for the first four modes of torsional vibration are given in these tables for various combinations of the parameters K, s and d.

It can be observed from Table 4.1 that in the case of simply-supported beams for K = 0.01, s = 0.10 and d = 0.05, the modifying quotients for the first four modes are respectively 0.944, 0.826, 0.705 and 0.603 and therefore the reductions in the first four torsional frequencies are respectively by 5.6%, 17.4%, 29.5% and39.7%. For K = 10.0, s = 0.10 and d = 0.05, the modifying quotients for the first four modes are respectively 0.986, 0.934, 0.851 and 0.762 and therefore the reductions in the first four torsional frequencies are respectively by 1.4%, 6.6%, 14.9% and 23.8%. From these values we can observe that the increase in the value of warping parameter K reduces the effects of longitudinal inertia and shear deformation on the torsional frequencies of vibration and that for smaller values

TABLE - 4.1

Effects of longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams. (a = 2d)

K	s	d	Values of frequency parameter λ^2 and modifying quotients $q = \lambda / \lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV MODE	q_4
0.01	0.00	0.00	97.411	1.000	1558.563	1.000	7890.216	1.000	24936.965	1.000
	0.04	0.02	95.559	0.990	1445.771	0.963	6724.678	0.923	19129.629	0.876
	0.08	0.04	90.361	0.963	1195.602	0.876	4747.525	0.776	11629.818	0.683
	0.10	0.05	86.882	0.944	1062.477	0.826	3920.978	0.705	9077.973	0.603
1.00	0.00	0.00	107.280	1.000	1598.038	1.000	7979.033	1.000	25094.863	1.000
	0.04	0.02	105.429	0.991	1484.710	0.964	6810.993	0.924	19281.160	0.877
	0.08	0.04	100.096	0.166	1233.507	0.879	4831.089	0.778	11777.295	0.685
	0.10	0.05	96.554	0.949	1099.974	0.830	4004.032	0.708	9225.332	0.606
10.00	0.00	0.00	1084.375	1.000	5506.419	1.000	16772.891	1.000	40728.391	1.000
	0.04	0.02	1078.618	0.997	5339.086	0.985	15368.900	0.957	34293.359	0.918
	0.08	0.04	1063.581	0.990	4982.955	0.951	13086.492	0.883	26296.227	0.804
	0.10	0.05	1053.563	0.986	4798.909	0.934	12158.678	0.851	23640.070	0.762

TABLE - 4.2

Effects of longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported thin-walled beams ($s=24$).

K	e	d	Values of the frequency parameter λ^2 and modifying quotients $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.01	0.00	0.00	249.614	1.000	3993.813	1.000	20218.664	1.000	63900.938	1.000
	0.04	0.02	243.820	0.988	3642.962	0.955	16690.797	0.909	46820.211	0.856
	0.08	0.04	227.685	0.955	2926.263	0.856	11414.037	0.751	27857.102	0.660
	0.10	0.05	217.290	0.933	2572.443	0.803	9390.227	0.681	21981.023	0.585
1.00	0.00	0.00	261.950	1.000	4043.156	1.000	20329.684	1.000	64098.313	1.000
	0.04	0.02	256.114	0.989	3693.813	0.956	16809.074	0.909	47040.188	0.857
	0.08	0.04	240.398	0.958	2981.270	0.859	11551.523	0.754	28131.027	0.662
	0.10	0.05	230.203	0.937	2630.307	0.807	9539 .969	0.685	22188.258	0.588
10.00	0.00	0.00	1483.319	1.000	8928.631	1.000	31322.004	1.000	83640.219	1.000
	0.04	0.02	1486.950	1.001	8727.613	0.989	28538.156	0.955	68848.235	0.907
	0.08	0.04	1499.197	1.005	8452.291	0.973	25412.449	0.901	56551.433	0.822
	0.10	0.05	1510.454	1.009	8451.986	0.973	25375.574	0.900	59883.250	0.846

T A B L E - 4.3

Effects of Longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped thin-walled Beams (s=2d).

K	s	d	Values of λ^2 and λ/λ_0							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.01	0.00	0.00	519.521	1.000	8312.322	1.000	42081.117	1.000	132997.094	1.000
	0.04	0.02	506.516	0.987	7553.774	0.953	34643.352	0.907	97904.031	0.858
	0.08	0.04	472.111	0.953	6119.002	0.858	24856.652	0.769	66324.172	0.706
	0.10	0.05	450.494	0.931	5463.667	0.811	21719.863	0.718	66035.985	0.705
1.00	0.00	0.00	532.679	1.000	8364.955	1.000	42199.539	1.000	133207.625	1.000
	0.04	0.02	520.175	0.988	7613.752	0.954	34796.836	0.908	98226.422	0.859
	0.08	0.04	487.097	0.956	6198.148	0.861	25105.473	0.771	67019.500	0.709
	0.10	0.05	466.436	0.936	5556.567	0.815	22061.781	0.723	63261.859	0.689
10.00	0.00	0.00	1835.473	1.000	13576.129	1.000	53924.686	1.000	154052.313	1.000
	0.04	0.02	1870.097	1.009	13551.494	0.999	52975.867	0.991	129726.219	0.918
	0.08	0.04	1973.504	1.037	14213.235	1.023	50029.805	0.963	84112 .531	0.739
	0.10	0.05	2054.938	1.058	15654.676	1.074	28024.945	0.721	15597.772	0.318

TABLE - 4.4

Values of the Second set of first ~~five~~ four torsional frequencies of simply supported thin-walled beams (s=2d).

-K	s	d	Values of second set of λ^2			
			I Mode	II Mode	III Mode	IV Mode
0.01	0.04	0.02	1593247.253	1684425.253	1833359.503	2036853.003
	0.08	0.04	105276.578	127303.313	162304.813	209397.000
	0.10	0.05	44847.953	58676.362	80492.469	109879.281
1.00	0.04	0.02	1593247.253	1684425.503	1833361.753	2036859.503
	0.08	0.04	105276.688	127304.875	162309 .969	209407.438
	0.10	0.05	44848.156	58678.828	80498.235	109889.797
10.00	0.04	0.02	1593251.003	1684479.753	1833597.753	2037480.503
	0.08	0.04	105290.313	127463.813	162848.407	210522.000
	0.10	0.05	44868.242	58888.274	81137.438	111108.594

TABLE - 4.5

Values of the Second set of first ~~five~~ four torsional frequencies of clamped-simply supported thin-walled beams (s=2d).

K	s	d	Values of Second set of λ^2			
			I Mode	II Mode	III Mode	IV Mode
0.01	0.04	0.02	1600809.503	1713070.503	1892789.253	2132533.007
	0.08	0.04	107066.906	133283.313	172987.188	224012.407
	0.10	0.05	45951.258	62101.703	86126.594	116815.516
1.00	0.04	0.02	1600809.503	1713069.003	1892782.003	2132510.506
	0.08	0.04	107066.531	133277.657	172960.719	223935.844
	0.10	0.05	45950.680	62093.180	86087.875	116705.656
10.00	0.04	0.02	1600800.253	1712920.753	1892045.003	2130244.506
	0.08	0.04	107029.125	132692.125	170092.125	215057.313
	0.10	0.05	45891.797	61156.977	81244.578	98552.562

T A B L E - 4.6

Values of the Second set of first four torsional frequencies of clamped-clamped thin-walled beams (s=2d).

K	s	d	Values of Second set of λ^2			
			I Mode	II Mode	III Mode	IV Mode
0.01	0.04	0.02	1603117.503	1719440.753	1897969.003	2122573.006
	0.08	0.04	107465.047	132660.813	165327.563	195826.219
	0.10	0.05	46129.297	60855.430	77498.063	79240.328
1.00	0.04	0.02	1603117.003	1719433.503	1897934.003	2122467.507
	0.08	0.04	107463.235	132634.282	165197.188	195541.469
	0.10	0.05	46126.516	60815.164	77274.578	82224.969
10.00	0.04	0.02	1603070.003	1718707.003	1894426.003	2111805.507
	0.08	0.04	107279.625	129830.344	149051.907	199093.094
	0.10	0.05	45840.797	55928.227	83036.547	150733.750

of K the reductions in the torsional frequencies at higher modes owing to these second order effects become quite significant and should be taken care of. Similar observations can be made from Tables 4.2 and 4.3 for clamped-simply supported and clamped-clamped beams. It can be also noticed that these reductions in the torsional frequencies due to longitudinal inertia and shear deformation are comparatively high in the case of clamped-clamped beams than in the case of clamped-simply supported or simply-supported beams.

The results for the second set of frequencies for the simply supported, clamped-simply supported and clamped-clamped beams are given in Tables 4.4, 4.5 and 4.6 respectively. It must be recalled here that these second set of frequencies exist solely due to the inclusion of these second order effects. From Tables 4.4 to 4.6, we observe that even in the case of second set, the effect of increase in the values of the parameters s and d is to reduce significantly the frequencies at higher modes of vibration. It is interesting to note that the increase in the value of the warping parameter K is having a negligible effect on these reductions in the frequencies of the second set for all the three boundary conditions considered here.

CHAPTER - V

FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS OF SHORT WIDE-FLANGED THIN-WALLED BEAMS INCLUDING THE EFFECTS OF LONGITUDINAL INERTIA AND SHEAR DEFORMATION*.

5.1. INTRODUCTION:

The problem of torsional vibrations of short wide-flanged thin-walled beams including the effects of longitudinal inertia and shear deformation is completely solved in Chapter IV utilising rigorous mathematical analysis. The highly transcendental frequency equations obtained for various end conditions could be solved only by lengthy trial-and-error procedure. Except for the case of simply-supported beam, the results for other complex boundary conditions could be obtained only by expending considerable effort.

Even the approximate analytical methods such as Ritz and Galerkin techniques have a tendency to become very tedious for some complex boundary conditions. The complexity of the analytical techniques even for simple end conditions emphasizes the need for physically satisfactory approximate solutions. To this end, the present Chapter aims at developing a finite element analysis of torsional vibrations of short wide-flanged thin-walled beams including the effects of longitudinal inertia and shear deformation.

---

* A paper by the author based on the results from this Chapter is accepted for publication in AIAA Journal, See Ref.(52).

The basic theory behind the finite element method for dynamic problems is briefly presented in Chapter III and is shown to give results which are in excellent agreement with the exact ones. This chapter, therefore, extends the finite element method to torsional vibrations of doubly-symmetric thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. New stiffness and mass matrices for a thin-walled beam are developed in this chapter, for the first time and, to the best of author's knowledge, there is no other finite element formulation for this problem available in the literature. The method developed in this chapter is applicable to uniform as well as non-uniform beams with any complex boundary conditions. A consistant mass matrix is made use of in conjunction with the corresponding stiffness matrix for finding the frequencies and mode shapes for free torsional vibrations of uniform thin-walled beams with various boundary conditions. Results obtained are compared with the exact ones obtained in Chapter IV and an excellent agreement is observed.

5.2. MODIFIED ENERGY EXPRESSIONS:

Two approaches are made to our present problem. In the first approach, the stiffness and mass matrices are developed in terms of the total angle of twist ϕ and the warping angle directly utilizing the strain and kinetic energy expressions (Eqs.4.12 and 4.13) derived in Chapter IV. By assuming only one degree of freedom for each of the angles ϕ and ψ, the stiffness and mass matrices each of 4 x 4 size are obtained which include the second order effects. But the matrices obtained in this approach, though not shown here, does not satisfy the exact boundary conditions and thus could not yield good results.

An alternative approach which will be discussed in detail in this chapter is to split the total angle of twist into two parts: One part is the twist calculated by neglecting the shear strain in the strain energy expression, (Eq.(4.12)); and the second part gives the contribution due to shear strain.

Let us define the total angle of twist ϕ as:

$$\phi(z,t) = \phi_t(z,t) + \phi_s(z,t) \tag{5.1}$$

where the subscript "t" denotes the part of the solution when the shear strain has been neglected, and the subscript s denotes the contribution of the shear strain to the total angle of twist. This type of choice has the advantage that when ϕ_s is equated to zero, the resulting expressions reduce back to the equations for the lengthy beams presented and solved in Chapter-II. This approach is quite convenient as it satisfactorily encompasses all boundary conditions of the present problem.

By substituting Eq.(5.1) into Eq.(4.9) we obtain:

$$u = (h/2)\ (\phi_t + \phi_s) \tag{5.2}$$

Substituting of Eq.(5.2) into Eq.(4.6) gives:

$$\psi + e_{sh} = \frac{h}{2}\frac{\partial\phi_t}{\partial z} + \frac{h}{2}\frac{\partial\phi_s}{\partial z} \tag{5.3}$$

From Eq.(5.3) we can write:

$$\psi = \frac{h}{2}\frac{\partial\phi_t}{\partial z} \tag{5.4}$$

and

$$\varepsilon_{sh} = \frac{h}{2} \frac{\partial \phi_s}{\partial z} \tag{5.5}$$

By substituting the expressions for ψ and ε_{sh} from Eqs.(5.4) and (5.5) respectively into Eqs.(4.4) and (4.7), the expressions for moment M and shear force Q can be obtained as:

$$M = EI_f \frac{h}{2} \frac{\partial^2 \phi_t}{\partial z^2} \tag{5.6}$$

and

$$- Q = K' A_f G \frac{h}{2} \frac{\partial \phi_s}{\partial z} \tag{5.7}$$

By substituting Eq.(5.1) into Eq.(4.1), the strain energy U_1 due to saint-venant torsion can be obtained as:

$$U_1 = \frac{1}{2} \int_0^L GC_s \left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z} \right)^2 dz \tag{5.8}$$

By substituting Eqs.(5.6) and (5.4) into Eq.(4.5), the strain energy U_2 of the two flanges due to warping normal strain becomes:

$$U_2 = \frac{1}{2} \int_0^L EC_w \left(\frac{\partial^2 \phi_t}{\partial z^2} \right)^2 dz \tag{5.9}$$

Substituting Eqs.(5.1) and (5.7) into Eqs.(2.2a) and (4.8), the expressions for the Saint-Venant torque T_s and the torque due to warping T_w can be respectively obtained as:

$$T_s = GC_s \left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z} \right) \tag{5.10}$$

and

$$T_w = - Qh = K' A_f G \frac{h^2}{2} \frac{\partial \phi_s}{\partial z} \tag{5.11}$$

Hence the total torque T_t (See Eq.4.10) can be obtained from Eqs.(5.10) and (5.11) as:

$$T_t = GC_s\left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right) + K'A_f G \frac{h^2}{2} \frac{\partial \phi_s}{\partial z} \tag{5.12}$$

Substituting Eqs.(5.7) and (5.5) into Eq.(4.11), the strain energy due to shear deformation of the two flanges, U_3, becomes:

$$U_3 = \frac{1}{2}\int_0^L K'A_f G \frac{h^2}{2}\left(\frac{\partial \phi_s}{\partial z}\right)^2 dz \tag{5.13}$$

The total strain energy, U, at any instant t (See Eq. 4.12) is the sum of the energies U_1, U_2 and U_3 and therefore given by

$$U = \frac{1}{2}\int_0^L \left[GC_s\left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right)^2 + EC_w\left(\frac{\partial^2 \phi_t}{\partial z^2}\right)^2 + K'A_f G \frac{h^2}{2}\left(\frac{\partial \phi_s}{\partial z}\right)^2 \right] dz \tag{5.14}$$

By substituting Eqs.(5.1) and (5.4) into Eq.(4.13), the total kinetic energy, T, at time t becomes:

$$T = \frac{1}{2}\int_0^L \left[\rho I_p\left(\frac{\partial \phi_t}{\partial t} + \frac{\partial \phi_s}{\partial t}\right)^2 + \rho C_w\left(\frac{\partial^2 \phi_t}{\partial z \partial t}\right)^2 \right] dz \tag{5.15}$$

5.3. MODIFIED NATURAL BOUNDARY CONDITIONS:

In terms of the angles ϕ_t and ϕ_s the natural boundary conditions given by Eqs.(4.19) to (4.22) can be modified as follows:

(a) Simply supported end:

$$\phi_s = 0; \quad \phi_t = 0; \quad \frac{\partial^2 \phi_t}{\partial z^2} = 0 \tag{5.16}$$

(b) Fixed end:

$$\phi_s = 0; \quad \phi_t = 0; \quad \frac{\partial \phi_t}{\partial z} = 0 \qquad (5.17)$$

(c) Free end:

$$\frac{\partial^2 \phi_t}{\partial z^2} = 0; \; GC_s \frac{\partial \phi_t}{\partial z} + (GC_s + K' A_f G\, h^2/2) \frac{\partial \phi_s}{\partial z} = 0 \qquad (5.18)$$

~~[illegible]~~ or

$$\frac{\partial \phi_t}{\partial z} = 0 \; ; \quad \frac{\partial \phi_s}{\partial z} = 0 \qquad (5.19)$$

The conditions given by Eq. ~~(5.18)~~ (5.19). are useful for finding symmetric modes of vibration in simply supported, fixed-fixed and free-free beams.

5.4. FINITE ELEMENT FORMULATION:

In the present formulation, for each finite element of a short thin-walled beam in torsion including the effects of longitudinal inertia and shear deformation in addition to warping, there are four generalized nodal displacements at the j end of the ith member. These nodal displacements are:

ϕ_{tj} = angle of twist neglecting shear strain at the shear center about z-axis;

ϕ'_{tj} = rate of change of ϕ_t at the shear center about z-axis;

ϕ_{sj} = angle of twist due to shear strain at the shear center about z-axis;

ϕ'_{sj} = rate of change of ϕ_s at the shear center about z-axis;

where subscript j denotes the generalized displacement at the j end of the ith finite element. Similar generalized nodal displacements exist at the K end of the element. The prime denotes differentiation with respect to z.

Assuming the angles ϕ_t and ϕ_s within each finite element to vary cubically the displacement functions take the form:

$$\phi_t(z) = a_1 + b_1 z + c_1 z^2 + d_1 z^3 \tag{5.20}$$

and

$$\phi_s(z) = a_2 + b_2 z + c_2 z^2 + d_2 z^3 \tag{5.21}$$

To establish relationships between the displacements at any interior coordinate z in terms of the generalized nodal coordinates, the eight arbitrary constants in the assumed displacement functions must be determined.

After determining the coefficients in Eqs.(5.20) and (5.21), the angles ϕ_t and ϕ_s at any coordinate z within the element in terms of the nodal displacements ϕ_{tj}, $\partial\phi_{tj}/\partial z$, ϕ_{tK}, and $\partial\phi_{tK}/\partial z$ and, ϕ_{sj}, $\partial\phi_{sj}/\partial z$, ϕ_{sK}, and $\partial\phi_{sK}/\partial z$ can be respectively defined as follows:

$$\phi_t(z) = \left[(1-3\bar{\xi}_1^2 + 2\bar{\xi}_1^3),\ z(1-2\bar{\xi}_1+\bar{\xi}_1^2),\ (3\bar{\xi}_1^2-2\bar{\xi}_1^3),\ z(-\bar{\xi}_1+\bar{\xi}_1^2)\right]\bar{R}_{tN}(t) \tag{5.22}$$

and

$$\phi_s(z)=\left[(1-3\bar{\zeta}_1^2+2\bar{\zeta}_1^3),z(1-2\bar{\zeta}_1+\bar{\zeta}_1^2),(3\bar{\zeta}_1^2-2\bar{\zeta}_1^3),z(-\bar{\zeta}_1+\bar{\zeta}_1^2)\right]\bar{R}_{sN}(t) \tag{5.23}$$

where $\bar{\zeta}_1 = z/l$.

Eqs.(5.22) and (5.23) can be written in an abbreviated form as follows:

$$\phi_t(z) = \bar{A}\,(z)\,\bar{R}_{tN}(t) \tag{5.24}$$

and

$$\phi_s(z) = \bar{A}\,(z)\,\bar{R}_{sN}(t) \tag{5.25}$$

where

$$\bar{R}_{tN}= [\,\phi_{tj},\,\phi'_{tj},\,\phi_{tK},\,\phi'_{tK}\,] \tag{5.26}$$

$$\bar{R}_{sN} = [\,\phi_{sj},\,\phi'_{sj},\,\phi_{sK},\,\phi'_{sK}\,] \tag{5.27}$$

and $\bar{A}\,(z)$ is given by Eq.(3.23).

Similarly, for the first and second derivatives of the angles ϕ_t and ϕ_s, the matrix relations can be written as:

$$\phi'_t(z) = (\bar{A}(z)\bar{R}_{tN}(t))'= \bar{A}_1(z)\bar{R}_{tN}(t) \tag{5.28}$$

$$\phi''_t(z) = (\bar{A}(z)\bar{R}_{tN}(t))''= \bar{A}_2(z)\bar{R}_{tN}(t) \tag{5.29}$$

$$\phi'_s(z) = (\bar{A}(z)\bar{R}_{sN}(t))'= \bar{A}_1(z)\bar{R}_{sN}(t) \tag{5.30}$$

and

$$\phi''_s(z) = (\bar{A}(z)\bar{R}_{sN}(t))''= \bar{A}_2(z)\bar{R}_{sN}(t) \tag{5.31}$$

where $\bar{A}_1(z)$ and $\bar{A}_2(z)$ are defined by Eqs.(3.27) and (3.28).

The generalized velocities and accelerations can also be expressed in terms of the discretized nodal velocities and accelerations:

That is:

$$\dot{\phi}_t(z) = \bar{A}(z)\,\dot{\bar{R}}_{tN}(t) \qquad (5.32)$$

$$\dot{\phi}'_t(z) = \bar{A}_1(z)\,\dot{\bar{R}}_{tN}(t) \qquad (5.33)$$

$$\ddot{\phi}_t(z) = \bar{A}(z)\,\ddot{\bar{R}}_{tN}(t) \qquad (5.34)$$

$$\dot{\phi}_s(z) = \bar{A}(z)\,\dot{\bar{R}}_{sN}(t) \qquad (5.35)$$

and

$$\ddot{\phi}_s(z) = \bar{A}(z)\,\ddot{\bar{R}}_{sN}(t) \qquad (5.36)$$

where dots denote differentiation with respect to time t.

5.5. Derivation of Element Matrices including Second Order Effects:

The expressions for the strain energy U, and Kinetic energy T_K, given by Eqs.(5.14) and (5.15) respectively, for an element of finite length, 1, can be written as follows:

$$U = \frac{1}{2}\int_0^l \left[GC_s(\phi'_t + \phi'_s)^2 + EC_w(\phi''_t)^2 + K'A_fG\frac{h^2}{2}(\phi'_s)^2\right]dz \qquad (5.37)$$

and

$$T = \frac{1}{2}\int_0^l \left[\rho I_p(\dot{\phi}_t + \dot{\phi}_s)^2 + \rho C_w(\dot{\phi}'_t)^2\right]dz \qquad (5.38)$$

Direct substitution of Eqs.(5.24) to (5.36) into Eqs.(5.37) and (5.38) and the resulting expressions into as eqns. representing Hamilton's Principle, Eq.(3.34) for W = 0 , yields (for the Nth element):

$$\delta I_N = \delta \int_{t_1}^{t_2} \left\{ \frac{\rho I_p}{2} \left[\int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{tN}\, dz + \int_0^1 \dot{\bar{R}}_{sN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{sN}\, dz \right.\right.$$

$$\left. + \int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{sN}\, dz + \int_0^1 \dot{\bar{R}}_{sN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{tN}\, dz \right]$$

$$+ \frac{\rho C_w}{2} \int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}_1^T \bar{A} \dot{\bar{R}}_{tN}\, dz$$

$$- \frac{1}{2} \int_0^1 \bar{R}_{tN}^T \left[EC_w \bar{A}_2^T \bar{A}_2 + GC_s \bar{A}_1^T \bar{A}_1 \right] \bar{R}_{tN}\, dz$$

$$- \frac{1}{2} \left(GC_s + K' A_f G \frac{h^2}{2}\right) \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{sN}\, dz$$

$$\left. - \frac{GC_s}{2} \left[\int_0^1 \bar{R}_{tN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{sN}\, dz + \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{tN}\, dz \right] \right\} dt$$

$$= 0 \qquad (5.39)$$

Eq.(5.39) can be also written more concisely as follows:

$$\delta I_N = \delta \int_{t_1}^{t_2} \frac{1}{2} \left[(\rho I_p L)\, \dot{\bar{q}}_N^T m_N \dot{\bar{q}}_N - (EC_w/L^3) \bar{q}_N^T \bar{K}_N \bar{q}_N \right] dt = 0 \qquad (5.40)$$

In Eq.(5.40) the terms $(\rho I_p L) m_N$ and $(EC_w/L^3)\bar{K}_N$ denote respectively the new mass and stiffness matrices $\bar{M}_N$ and $\bar{K}_N$ respectively of the Nth element. The matrices $\bar{m}_N$, $\bar{K}_N$ and $\bar{q}_N$ are given below:

$$m_N = \cancel{\frac{1}{420\, N^4}} \begin{bmatrix} \bar{m}_{11} & \bar{m}_{21}^T \\ \bar{m}_{21} & \bar{m}_{22} \end{bmatrix} \qquad (5.41)$$

$$\bar{K}_N = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{21}^T \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix} \qquad (5.42)$$

and

$$\bar{q}_N = [\bar{q}_{tN}, \ \bar{q}_{sN}] \qquad (5.43)$$

where

$$\bar{m}_{11} = \frac{1}{420N^4} \begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} + \frac{d^2N^2}{30} \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.44)$$

$$\bar{m}_{21} = \bar{m}_{22} = \frac{1}{420N^4} \begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (5.45)$$

$$\bar{K}_{11} = \begin{bmatrix} 12N^2 & & \text{Sym.} & \\ 6N & 4 & & \\ -12N^2 & -6N & 12N^2 & \\ 6N & 2 & -6N & 4 \end{bmatrix}$$

$$+ \frac{K^2}{30N^2} \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.46)$$

$$\bar{K}_{21} = \frac{K^2}{30N^2} \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.47)$$

$$\bar{K}_{22} = \frac{(s^2K^2+1)}{30\, s^2N^2} \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.48)$$

$$\bar{q}_{tN} = [\, \phi_{tj},\ L\phi'_{tj},\ \phi_{tK},\ L\phi'_{tK} \,] \qquad (5.49)$$

$$\bar{q}_{sN} = [\, \phi_{sj},\ L\phi'_{sj},\ \phi_{sK},\ L\phi'_{sK} \,] \qquad (5.50)$$

and the non-dimensional parameters K^2, d^2 and s^2 are previously defined by Eqs.(4.39), (4.40), and (4.41) respectively.

The equations of motion for the discretized system can now be obtained using Eq.(5.40). Taking the variation of the integral expression of Eq.(5.40) we obtain:

$$\int_{t_1}^{t_2} [\, (\rho I_p L)\, \delta\, \dot{\bar{q}}_N^T\, \bar{m}_N\, \dot{\bar{q}}_N - (EC_w/L^3)\, \delta\, \bar{q}_N^T\, \bar{K}_n\, \bar{q}_N \,]\, dt = 0 \qquad (5.51)$$

which after integration by parts over the time interval gives:

$$(\rho I_p L)\,\bar{\delta}\,\bar{q}_N^T\,\bar{m}_N\,\bar{q}_N\Big|_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2}\bar{\delta}\,\bar{q}_N^T\left[(\rho I_p L)\,\bar{m}_N\,\ddot{\bar{q}}_N + (EC_w/L^3)\,\bar{K}_N\,\bar{q}_N\right]dt = 0 \qquad (5.52)$$

The first term in Eq.(5.52) is seen to vanish in view of the assumptions made previously that the virtual displacements $\bar{\delta}\bar{q}_N$ are zero at the time instants t_1 and t_2. Since the virtual displacements can be arbitrary for other times then the only way in which the integral expression in Eq.(5.52) can vanish is for the terms within the brackets to equal zero. Therefore, the governing dynamic equilibrium equations for the discretized systems are:

$$(\rho I_p L)\bar{m}_N\ddot{\bar{q}}_N + (EC_w/L^3)\,\bar{K}_N\bar{q}_N = 0 \qquad (5.53)$$

Assuming that the displacements undergo harmonic oscillation, the displacement vector $\bar{q}_N$ can be written as:

$$q_N = \bar{Q}_N\,e^{ip_n t} \qquad (5.54)$$

where $\bar{Q}_N$ is a column vector of torsional amplitudes of the general torsional displacements. Substituting Eq.(5.54) into (5.53) gives:

$$\left[\,(EC_w/L^3)\bar{K}_N - (\rho I_p L\,p_n^2)\bar{m}_N\right]\bar{Q}_N\,e^{ip_n t} = 0 \qquad (5.55)$$

Deviding throughout by EC_w/L^3 and cancelling e^{ip_nt}, Eq.(5.55) becomes

$$[\bar{K}_N][\bar{Q}_N] = \lambda^2 [\bar{m}_N][\bar{Q}_N] \qquad (5.56)$$

where λ^2 is the non-dimensional frequency parameter defined previously by (Eq.(4.38). Eq.(5.56) represents the equations of motion for an undamped free oscillating system including the effects of longitudinal inertia and shear deformation.

5.6. Equations of Equilibrium for the totally assembled beam:

Following the procedure outlined in section 3.5 and utilising the element stiffness and mass matrices presented in section 5.5, the equations of equilibrium for the totally assembled beam can be obtained as:

$$[\bar{K}][\bar{Q}] = \lambda^2 [\bar{m}][\bar{Q}] \qquad (5.57)$$

where $\bar{K}$, $\bar{m}$ and $\bar{Q}$ denote the totally assembled matrices corresponding to the element matrices $\bar{K}_N$, $\bar{m}_N$ and $\bar{Q}_N$ defined previously. With the four generalized displacements possible at each node and with the bar segmented into N elements, the total number of degrees of freedom is 4 (N+1). The formulation of the matrix equilibrium equation, Eq.(5.57), includes all possible degrees of freedom, both free and restrained. The displacement vector Q of this overall joint equilibrium equations is comprized of both degrees of freedom, the unknowns of the problem and known support displacements or boundary conditions.

5.7. Boundary conditions useful for Modifying the total Matrices:

It should be recalled here that for the present finite element formulation, totally four generalized displacements are considered at each node. The following are therefore the boundary conditions to be utilized in order to modify the total stiffness and mass matrices for various combinations of end supports.

(a) Simply supported end:

$$\phi_s = 0 \; ; \; \phi_t = 0 \qquad (5.58)$$

(b) Fixed end:

$$\phi_s = 0 \; ; \; \phi_t = 0 \; ; \; L\phi_t' = 0 \qquad (5.59)$$

(c) Free end:

The total matrices need not be modified in this case.

(d) $$L\,\phi_t' = 0 \; ; \; L\,\phi_s' = 0 \qquad (5.60)$$

Eqs. (5.58) to (5.60) are useful for finding symmetric modes of vibration in simply supported, fixed-fixed and free-free beams.

5.8. RESULTS AND CONCLUSIONS:

A digital computer programme is written in Fortran IV which can give results for any set of boundary conditions. Results for simply supported and fixed-fixed beams for values of K = 1.541, s = 0.046 and d = 0.023, are obtained on IBM 1130 Computer at Andhra University, Waltair and are presented in Tables 5.1 and 5.2.

For the simply supported case, the first and second sets of values of λ obtained for the first four modes of vibration for a division of the beam into N = 2 and 3 segments are shown in Table 5.1 and are compared with the exact results obtained using the analysis presented in Chapter IV. For, the fixed-fixed beam, the first set of values of λ obtained for the first four modes of vibration of N = 2 and 3 are shown in Table 5.2 and are compared with the exact results. The exact results for the simply supported case were obtained using Eq.(4.65) and for the fixed-fixed beam, the results were obtained using Eqs.(4.44) and (4.72).

It can be seen from Tables 5.1 and 5.2 that for all cases, excellent results have been obtained even for very coarse subdivisions of the beam. Since the stiffness and mass matrices including shear deformation and longitudinal inertia seperately involve double the number of degrees of freedom than those that exist if they are neglected, twice as many natural frequencies result. In Table 5.1 the lower and higher spectrum of frequen-

TABLE - 5.1

Comparison of first and second sets of values of λ from the Finite element Method and those from exact analysis given in Chapter IV for a simply supported beam (K=1.541, s=0.046, d=0.023).

Mode	Exact Values of λ from Chap.IV	No.of elements and % error					
		One element	% error	Two elements	% error	Three elements	% error
First Set:							
I	10.8722	11.7421	8.01%	11.1132	2.2%	10.8814	0.08%
II	38.7942	47.9234	23.54%	42.2221	8.84%	38.9231	0.33%
III	81.3913			108.1012	32.82%	96.9422	19.10%
IV	134.8025			161.4034	19.73%	151.3014	12.24%
V	195.6023					240.7015	23.06%
Second Set:							
I	962.54	964.72	0.23%	963.44	0.09%	962.73	0.02%
II	998.22	1018.43	2.03%	1007.25	0.90%	999.35	0.11%
III	1053.37			1093.14	3.78%	1072.06	1.78%
IV	1124.52			1191.38	5.93%	1165.17	3.60%
V	1207.32					1317.43	

TABLE - 5.2

Comparison of the first set of values of λ from the finite element method and those from exact analysis given in Chapter IV for a fixed-fixed beam (K=1.541, s= 0.046, d=0.023).

Mode	Exact Values of λ from Chap.IV	No.of elements and % error			
		Two elements	% error	Three elements	% error
I	21.6699	21.8663	0.91%	21.8374	0.78%
II	55.9769	69.3964	23.94%	67.8850	21.24%
III	101.7908	185.9526	82.96%	116.5183	14.47%
IV	155.7791	241.3891	54.96%	194.7396	25.01%
V	215.4931			303.6783	40.93%

cies obtained can also be observed to be in excellent agreement with the exact ones. In Chapter IV, we have discussed this second set of frequencies in detail.

Using the above stiffness and mass matrices, beams with various other boundary conditions, can be analyzed easily. A beam with variable cross section can also be analyzed by dividing the beam into a number of segments and assuming that each segment has a constant cross section. In all cases (as we observed from Tables 5.1 and 5.2), the method gives an upper bound to the exact frequencies of the system. The approach presented in the Chapter is quite general, satisfactorily encompasses all boundary conditions and can be extended to static and dynamic stability of uniform and tapered thin-walled beams.

CHAPTER - VI

FORCED TORSIONAL VIBRATIONS OF SHORT WIDE-FLANGED BEAMS WITH LONGITUDINAL INERTIA, SHEAR DEFORMATION AND VISCOUS DAMPING*.

6.1. INTRODUCTION:

In Chapters IV and V, the problem of free torsional vibrations of short thin-walled beams of open section, including the effects of longitudinal inertia and shear deformation is completely analyzed utilizing the exact and approximate analytical methods and the powerful finite-element technique.

With regards to the forced torsional vibrations of thin-walled beams of open section very few studies are available in the literature. Tso (104), extended the Timoshenko torsion theory for coupled flexural-torsional vibrations of thin-walled beams of open sections and presented a formal solution to Gere's theory (32) under general loading conditions and general boundary conditions. Aggarwal (3), considered the problem of forced torsional vibrations of thin-walled beams of open section under very general loads including the effects of longitudinal inertia and shear deformation, and solved the specific case of a simply supported beam with a step torque impulsively applied at the mid-point. He compared the results obtained for the above problem, with those obtained utilizing Timoshenko torsion theory. But in all these studies the effect of damping has not considered.

---

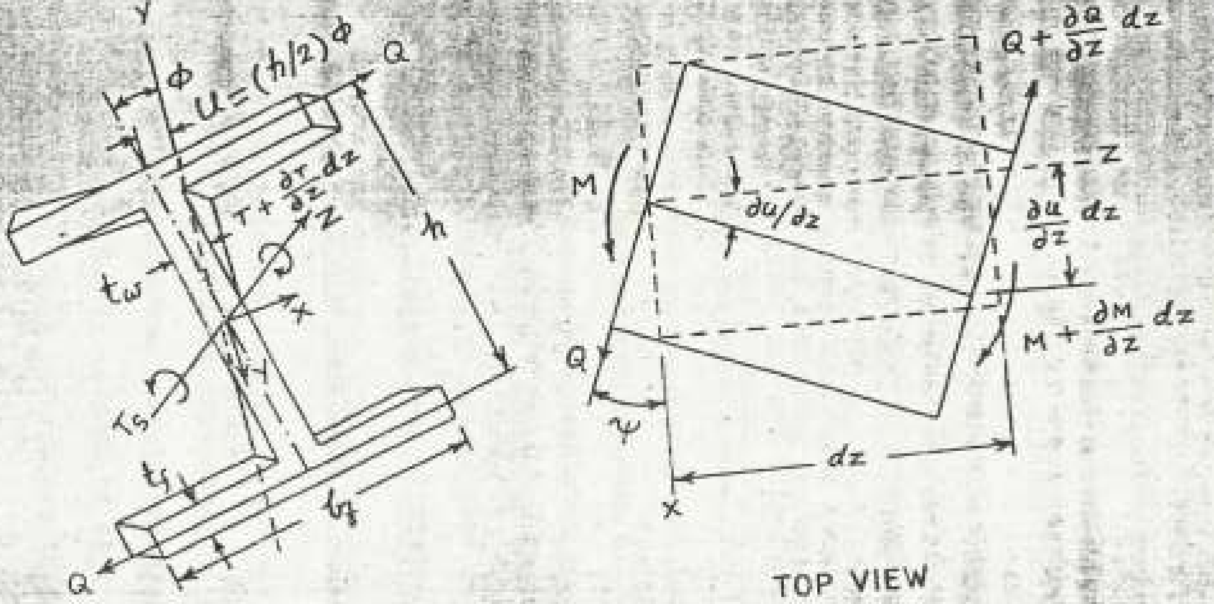
* A paper by the author, abstracted from this Chapter, is accepted for publication in the August 1976 issue of the Journal of the Aeronautical Society of India. See Ref. (53)

The present Chapter therefore deals with the study of forced torsional vibrations of doubly-symmetric thin-walled beams of open section such as an I-beam, including the effects of longitudinal inertia, shear deformation and viscous damping. Viscous damping forces arising separately from torsional and warping velocities are included in the equations of motion and the coupled fundamental equations of motion are formulated in terms of angle of twist and warping angle. The method of solution is demonstrated for arbitrary external torque for the beam having both ends simply-supported and numerical results are presented for the case when the torque is uniform over the span and varies sinusoidally in time. Amplitude response is plotted versus torsional frequency for varying amounts of torsional and warping damping, and is compared to the response for the classical beam (based on Timoshenko torsion theory) for the first five symmetric mode shapes.

6.2. DERIVATION OF EQUATIONS OF MOTION INCLUDING VISCOUS DAMPING:

In Fig.6.1, a typical differential element of length dz and width b_f is taken from the flange of the thin-walled beam, and the generalized forces acting are shown. Assuming small displacements as in Chapter IV and summing the torques yields one equation of motion:

$$\frac{\partial}{\partial z}\left(T_s + T_w\right) - \beta_t \frac{\partial \phi}{\partial t} + T_e = \rho I_p \frac{\partial^2 \phi}{\partial t^2} \qquad (6.1)$$


Y
ϕ
$u = (h/2)\phi$
Q
$\tau + \frac{\partial \tau}{\partial z} dz$
Z
h
t_w
X
Y
τ_s
t_f
b_f
Q
$Q + \frac{\partial Q}{\partial z} dz$
M
Z
$\partial u/\partial z$
$\frac{\partial u}{\partial z} dz$
$M + \frac{\partial M}{\partial z} dz$
Q
ψ
dz
X

FRONT VIEW

TOP VIEW

FIG. 6·1. STRAINED STATE OF A BEAM ELEMENT where T_s is the Saint Venant torque given by Eq.(2.2a), T_w the warping torque given by Eq.(4.8), β_t the torsional damping constant and, T_e the external torque per unit length of the beam.

Summing moments about an axis normal to Fig.6.1 yields the second equation of motion:

$$\frac{\partial M}{\partial z} - Q - qb_f = \rho I_f \frac{\partial^2 \psi}{\partial t^2} \qquad (6.2)$$

where M is the bending moment in the top flange given by Eq.(4.4), Q the shear force given by Eq.(4.7), q the external viscous force per unit length acting along the sides of the flanges, of width b, to oppose warping.

Further, let us define a warping damping constant β_w by:

$$q = \frac{\beta_w}{b_f} \frac{\partial \psi}{\partial t} \qquad (6.3)$$

Substituting Eqs.(2.2a), (4.8), (4.4), (4.7) and (6.3) in Eqs.(6.1) and (6.2) we obtain:

$$GC_s \frac{\partial^2 \phi}{\partial z^2} + K' A_f Gh\left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z}\right) + T_e = \rho I_p \frac{\partial^2 \phi}{\partial t^2} + \beta_t \frac{\partial \phi}{\partial t} \qquad (6.4)$$

and

$$EI_f \frac{\partial^2 \psi}{\partial t^2} + K' A_f G\left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi\right) = \rho I_f \frac{\partial^2 \psi}{\partial t^2} + \beta_w \frac{\partial \psi}{\partial t} \qquad (6.5)$$

It is necessary to obtain solutions to the differential Equations (6.4) and (6.5) which also satisfy the boundary conditions of the particular problem being considered. This may be achieved by assuming solutions in the form:

$$\phi(z, t) = \sum_{n} \bar{\phi}_n(z)\, F_n(t) \tag{6.6}$$

$$\psi(z, t) = \sum_{n} \bar{\psi}_n(z)\, G_n(t) \tag{6.7}$$

where $\bar{\phi}_n(z)$ and $\bar{\psi}_n(z)$ are the mode shapes obtained from solving the free, undamped vibration problem. The mode shape functions are given in section 4.7 of Chapter IV for the six cases arising from combinations of simply supported, clamped and free ends. This procedure will be used below to investigate the case when both ends are simply supported.

6.3. SOLUTION FOR THE CASE OF A SIMPLY SUPPORTED BEAM:

Consider a beam of length L having its ends z=0 and z=L both simply supported. From Eq.(4.65) of Chapter IV, the frequencies of vibration for this case are given in an alternative form as:

$$p_n^2 = \frac{-\bar{b} \pm (\bar{b}^2 - 4\bar{a}\bar{c})^{1/2}}{2a} \tag{6.8}$$

where

$$a = \frac{\rho I_p \rho I_f L^4}{K' A_f G} \tag{6.9}$$

$$\bar{b} = -\left[\rho I_p L^4 + n^2\pi^2 L^2 \left(\frac{\rho I_p I_f}{K' A_f G}\right) + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}) \right] \tag{6.10}$$

$$\bar{c} = n^2\pi^2 L^2 G C_s + n^4\pi^4 \left(\frac{E I_f C_s}{K' A_f} + E C_w\right) \tag{6.11}$$

From Eqs.(4.67) and (4.68) of Chapter IV, the mode shapes for this case are given by:

$$\bar{\phi}_n(z) = A_n \sin \frac{n\pi z}{L} \tag{6.12}$$

$$\bar{\psi}_n(z) = B_n \cos \frac{n\pi z}{L} \tag{6.13}$$

where A_n and B_n are arbitrary amplitudes.

Let the external torque per unit length be expressed as:

$$T_e(z, t) = \sum_{n=1}^{\infty} \tau_n(t) \sin \frac{n\pi z}{L} \tag{6.14}$$

where Fourier coefficients are determined from

$$\tau_n(t) = \frac{2}{L} \int_0^L T_e(z,t) \sin \frac{n\pi z}{L}\, dz \tag{6.15}$$

The solution of the coupled differential Eqs.(6.4) and (6.5) can progress in several ways. We will begin by first uncoupling them. Differentiating Eq.(6.4) with respect to z, solving Eq.(6.4) for $\partial\psi/\partial z$, and its higher derivatives, and substituting into Eq.(6.5) yields a fourth order uncoupled equation for ϕ given by:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w\right] \frac{\partial^4 \phi}{\partial z^4} - \left[\frac{E\rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right] \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- GC_s \frac{\partial^2 \phi}{\partial z^2} - \left[\frac{EI_f \beta_t}{K' A_f G} + \frac{\beta_w C_s}{K' A_f} + \frac{\beta_w h^2}{2}\right] \frac{\partial^3 \phi}{\partial z^2 \partial t}$$

$$+ \frac{\rho^2 I_p I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial t^4} + \left[\frac{\rho I_f \beta_t}{K' A_f G} + \frac{\rho I_p \beta_w}{K' A_f G}\right] \frac{\partial^3 \phi}{\partial t^3} + \left[\rho I_p + \frac{\beta_t \beta_w}{K' A_f G}\right] \frac{\partial^2 \phi}{\partial t^2}$$

$$+ \beta_t \frac{\partial \phi}{\partial t} = T_e + \frac{1}{K' A_f G} \left[- E I_f \frac{\partial^2 T_e}{\partial z^2} + \rho I_f \frac{\partial^2 T_e}{\partial t^2} + \beta_w \frac{\partial T_e}{\partial t}\right] \qquad (6.16)$$

Similarly, eliminating ϕ between Eqs.(6.4) and (6.5) yields the uncoupled equation for ψ given by:

$$\left[\frac{E I_f C_s}{K' A_f} + E C_w\right] \frac{\partial^4 \psi}{\partial z^4} - \left[\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right] \frac{\partial^4 \psi}{\partial z^2 \partial t^2}$$

$$- G C_s \frac{\partial^2 \psi}{\partial z^2} - \left[\frac{E I_f \beta_t}{K' A_f G} + \frac{\beta_w C_s}{K' A_f} + \frac{\beta_w h^2}{2}\right] \frac{\partial^3 \psi}{\partial z^2 \partial t}$$

$$+ \frac{\rho^2 I_p I_f}{K' A_f G} \frac{\partial^4 \psi}{\partial t^4} + \left[\frac{\rho I_f \beta_t}{K' A_f G} + \frac{\rho I_p \beta_w}{K' A_f G}\right] \frac{\partial^3 \psi}{\partial t^3}$$

$$+ \left[\rho I_p + \frac{\beta_t \beta_w}{K' A_f G}\right] \frac{\partial^2 \psi}{\partial t^2} + \beta_t \frac{\partial \psi}{\partial t} = \frac{h}{2} \frac{\partial T_e}{\partial z} \qquad (6.17)$$

As expected, the left-hand sides of Eqs.(6.16) and (6.17) are identical.

Substituting Eqs.(6.6), (6.7), (6.12), (6.13) and (6.14) into Eqs.(6.16) and (6.17) results in:

$$\left[\frac{n^4\pi^4}{L^4}\left(\frac{EI_fC_s}{K'A_f}+EC_w\right\}+\frac{n^2\pi^2GC_s}{L^2}\right]F_n(t)$$

$$+\left\{\beta_t+\frac{n^2\pi^2}{L^2}\left(\frac{EI_f\beta_t}{K'A_fG}+\frac{\beta_wC_s}{K'A_f}+\frac{\beta_wh^2}{2}\right)\right\}\dot{F}_n(t)$$

$$+\left\{\rho I_p+\frac{\beta_t\beta_w}{K'A_fG}+\frac{n^2\pi^2}{L^2}\left(\frac{E\rho I_pI_f}{K'A_fG}+\frac{C_s\rho I_f}{K'A_f}+\frac{\rho I_fh^2}{2}\right)\right\}\ddot{F}_n(t)$$

$$+\left(\frac{\rho I_f\beta_t}{K'A_fG}+\frac{\rho I_p\beta_w}{K'A_fG}\right)\dddot{F}_n(t)+\frac{\rho^2I_pI_f}{K'A_fG}\ddddot{F}_n(t)$$

$$=\left(1+\frac{n^2\pi^2EI_f}{K'A_fGL^2}\right)\tau_n(t)+\frac{\beta_w}{K'A_fG}\dot{\tau}_n(t)+\frac{\rho I_f}{K'A_fG}\ddot{\tau}_n(t) \qquad (6.18)$$

$$\left\{\frac{n^4\pi^4}{L^4}\left(\frac{EI_fC_s}{K'A_f}+EC_w\right)+\frac{n^2\pi^2GC_s}{L^2}\right\}G_n(t)$$

$$+\left\{\beta_t+\frac{n^2\pi^2}{L^2}\left(\frac{EI_f\beta_t}{K'A_fG}+\frac{\beta_wC_s}{K'A_f}+\frac{\beta_wh^2}{2}\right)\right\}\dot{G}_n(t)$$

$$+\left\{\rho I_p+\frac{\beta_t\beta_w}{K'A_fG}+\frac{n^2\pi^2}{L^2}\left(\frac{E\rho I_pI_f}{K'A_fG}+\frac{C_s\rho I_f}{K'A_f}+\frac{\rho I_fh^2}{2}\right)\right\}\ddot{G}_n(t)$$

$$+\left(\frac{\rho I_f\beta_t}{K'A_fG}+\frac{\rho I_p\beta_w}{K'A_fG}\right)\dddot{G}_n(t)+\frac{\rho^2I_pI_f}{K'A_fG}\ddddot{G}(t)=\frac{n\pi h}{2L}\tau_n(t) \qquad (6.19)$$

where dots denote differentiations with respect to time. Eqs.(6.18) and (6.19) contain an exciting torsional function $\tau_n(t)$ which can be of any form.

6.4. RESPONSE TO A UNIFORMLY DISTRIBUTED TORSIONAL FORCING FUNCTION SINUSOIDAL IN TIME:

For purposes of detailed numerical results, let $T_e(z,t)$ be

$$T_e(z,t) = T_o \sin \omega t \tag{6.20}$$

where T_o is a constant and ω the torsional excitation frequency. Then, from Eq.(6.16) it follows that:

$$\tau_n(t) = \frac{4T_o}{n\pi} \sin \omega t, \quad n = 1,3,5,\ldots \tag{6.21}$$

Assuming a solution in the form

$$F_n(t) = A_n \sin \omega t + B_n \cos \omega t \tag{6.22}$$

Substituting Eqs.(6.21) and (6.22) into Eq.(6.18), and equating coefficients of $\sin \omega t$ and $\cos \omega t$ yields

$$A_n = \frac{4\,T_o \left\{ K_{1n}\left[K' A_f G + (n^2\pi^2/L^2)EI_f - \rho I_f \omega^2\right] + K_{2n}\beta_w \omega \right\}}{n\pi K' A_f G \left(K_{1n}^2 + K_{2n}^2 \right)} \tag{6.23}$$

$$B_n = \frac{4\,T_o \left\{ K_{1n}\beta_w \omega - K_{2n}\left[K' A_f G + (n^2\pi^2/L^2)EI_f - \rho I_f \omega^2\right] \right\}}{n\pi K' A_f G \left(K_{1n}^2 + K_{2n}^2 \right)} \tag{6.24}$$

where

$$K_{1n}=\left\{\left[\frac{n^4\pi^4}{L^4}\left(\frac{EI_fC_s}{K'A_f}+EC_w\right)+\frac{n^2\pi^2GC_s}{L^2}\right]\right.$$

$$-\left[\rho I_p+\frac{\beta_t\beta_w}{K'A_fG}+\frac{n^2\pi^2}{L^2}\left(\frac{E\rho I_pI_f}{K'A_fG}+\frac{C_s\rho I_f}{K'A_f}+\frac{\rho I_fh^2}{2}\right)\right]\omega^2$$

$$\left.+\ \frac{\rho^2I_pI_f}{K'A_fG}\,\omega^4\right\} \qquad (6.25)$$

$$K_{2n}=\left\{\omega\beta_t\left(1+\frac{n^2\pi^2EI_f}{K'A_fGL^2}\right)+\omega\beta_w\,\frac{n^2\pi^2}{L^2}\left(\frac{C_s}{K'A_f}+\frac{h^2}{2}\right)\right.$$

$$\left.-\ \frac{\omega^3\rho}{K'A_fG}\left(\beta_tI_f+\beta_wI_p\right)\right\} \qquad (6.26)$$

Similarly, assuming a solution

$$G_n(t)=C_n\sin\omega t+D_n\cos\omega t \qquad (6.27)$$

and substituting Eq.(6.21) and (6.27) into Eq.(6.19) yields:

$$C_n=\frac{2T_oh\,K_{1n}}{L(K_{1n}^2+K_{2n}^2)}\ ;\quad D_n=\frac{-2T_oh\,K_{2n}}{L(K_{1n}^2+K_{2n}^2)} \qquad (6.28)$$

where K_{1n} and K_{2n} are defined by Eqs.(6.25) and (6.26).

Of course, Eqs.(6.22) and (6.27) may be replaced in a more convenient phase angle form as:

$$F_n(t) = \sqrt{A_n^2 + B_n^2}\,\sin(\omega t + \text{arc tan } B_n/A_n) \qquad (6.29)$$

$$G_n(t) = \sqrt{C_n^2 + D_n^2}\,\cos(\omega t + \text{arc tan } D_n/C_n) \qquad (6.30)$$

Further we note that $D_n/C_n = - B_n/A_n$

6.5. FREE AND FORCED VIBRATIONS OF A CLASSIC BEAM SIMPLY SUPPORTED AT BOTH ENDS:

For purposes of comparing with the preceding results, let us now summarize the classic solution. In the case of the classic beam based on Timoshenko torsion theory, the effects of longitudinal inertia and shear deformation are neglected and by putting $1/K' = 0$ and $\rho I_f = 0$ in Eq.(6.16) we obtain:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} + \beta_t \frac{\partial \phi}{\partial t} = T_s \qquad (6.31)$$

Considering first, free vibrations with no damping, the differential equation becomes

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (6.32)$$

which was treated in detail by Gere (32).

The solution to this equation in terms of circular and hyperbolic functions is well known (32). It can be seen that a function which satisfies the boundary conditions of a beam simply supported at both ends is given by:

$$\phi = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi z}{L} \qquad (6.33)$$

Substituting Eq.(6.33) into Eq.(6.32) and recognizing that the resulting equation must be satisfied for all values of z within $0 \leq z \leq L$ gives

$$\rho I_p \ddot{F}_n(t) + \frac{n^2\pi^2}{L^2}\left(\frac{n^2\pi^2 EC_w}{L^2} + GC_s\right) F_n(t) = 0 \qquad (6.34)$$

From Eq.(6.34), the well known (32) frequency equation is found to be:

$$p_n = \frac{n\pi}{\sqrt{\rho}L}\left[\frac{n^2\pi^2 EC_w + L^2 GC_s}{I_p L^2}\right]^{1/2} \qquad (6.35)$$

For the steady-state solution of the forced, damped vibration problem as before, assume

$$\phi = \sum_{n=1}^{\infty} F_n(t) \sin\frac{n\pi z}{L} \qquad (6.36)$$

$$T_s(z,t) = \sum_{n=1}^{\infty} \tau_n(t) \sin\frac{n\pi z}{L} \qquad (6.37)$$

where, from Eq.(6.15)

$$\tau_n(t) = \frac{4T_o}{n\pi} \sin\omega t, \ (n=1,3,5,\ldots) \qquad (6.38)$$

Substituting Eqs.(6.36), (6.37) and (6.38) into Eq.(6.31) yields

$$\frac{n^2\pi^2}{L^2}\left[\frac{n^2\pi^2}{L^2} EC_w + GC_s\right] F_n(t) + \beta_t \dot{F}_n(t) + \rho I_p \ddot{F}_n(t) = \frac{4T_o}{n\pi}\sin\omega t \qquad (6.39)$$

having a steady-state solution

$$F_n(t) = E_n \sin\omega t + H_n \cos\omega t \qquad (6.40)$$

Substituting Eq.(6.40) into Eq.(6.39), we obtain

$$E_n = \frac{(4T_o/n\pi)\{(n^2\pi^2/L^2)\left[(n^2\pi^2/L^2)EC_w + GC_s\right] - \omega^2 \rho I_p\}}{\{(n^2\pi^2/L^2)\left[(n^2\pi^2/L^2)EC_w + GC_s\right] - \omega^2 \rho I_p\}^2 + (\beta_t \omega)^2} \qquad (6.41)$$

$$H_n = \frac{-(4T_o\ \beta_t \omega/n\pi)}{(n^2\pi^2/L^2)\left[(n^2\pi^2/L^2)EC_w + GC_s\right] - \omega^2 \rho\ I_p\}^2 + (\beta_t \omega)^2} \qquad (6.42)$$

or

$$F_n(t) = \frac{4T_o}{n\pi}\left\{\rho^2 I_p^2 (p_n^2 - \omega^2)^2 + (\beta_t \omega)^2\right\}^{1/2} \sin(\omega t + \theta) \qquad (6.43)$$

where

$$\tan\theta = \frac{-\beta_t \omega}{\rho I_p (p_n^2 - \omega^2)} \qquad (6.44)$$

6.6. DISCUSSION OF NUMERICAL RESULTS:

The solutions obtained were programmed on IBM-1130 Computer at Andhra University, Waltair, to allow a numerical study of the effects of the parameters involved. Some of the interesting results obtained are shown in Figs.6.2 to 6.8. In Figs.6.2 to 6.8, only the response of the first mode shape is considered. The values of the constants used for these figures are as follows:

$n=1$; $\rho = 0.00884332 (\text{lbs/in}^3)$; $E = 30 \times 10^6 (\text{lbs/in}^2)$;

$G = 12 \times 10^6 (\text{lbs/in}^2)$; $A_f = 20.7584 (\text{in}^2)$; $I_f = 469.532 (\text{in}^4)$;

$I_p = 17245.7 (\text{in}^4)$; $C_s = 27.3252 (\text{in}^4)$; $C_w = 3{,}02{,}231 (\text{in}^6)$;

$L = 760 (\text{in})$ and $T_o = 1.0$,

which correspond to a wide-flanged steel I-beam, 36 WF 230, with width of the flanges b = 16.475(in), height between the center lines of the flanges h = 35.88(in), thickness of the web t =0.765 (in) and thickness of the flanges t_f= 1.26(in).

Fig.6.2 is the plot of torsional amplitude against forcing function frequency with varying values of torsional damping for the classical beam based on Timoshenko torsion theory.

Figs.6.3, 6.4 and 6.5 are the plots of amplitude versus frequency including the effects of longitudinal inertia and shear deformation. For each set of the curves, the value of β_w, the damping associated with warping angle, is held constant while the values of torsional damping β_t are varied.

It can be observed that the general shapes of the plots do not differ at all from that of Fig.6.2, i.e., shear deformation and longitudinal inertia effects do not radically alter the form of the amplitude-frequency curves. As expected, increasing the damping associated with warping angle has the effect of lowering the amplitudes.

Figs.6.6, 6.7 and 6.8 are also amplitude frequency plots including longitudinal inertia and shear deformation effects, but for each set of curves β_t is held constant while β_w is varied from zero to 10^5. Again, the general form of the curves is not unlike that for the classical beam. However, comparing Figs.6.6, 6.7 and 6.8 with Figs.6.3, 6.4 and 6.5, it will be readily seen that the variation of damping associated with angle of twist β_t, has a much stronger influence on the curves than the variation

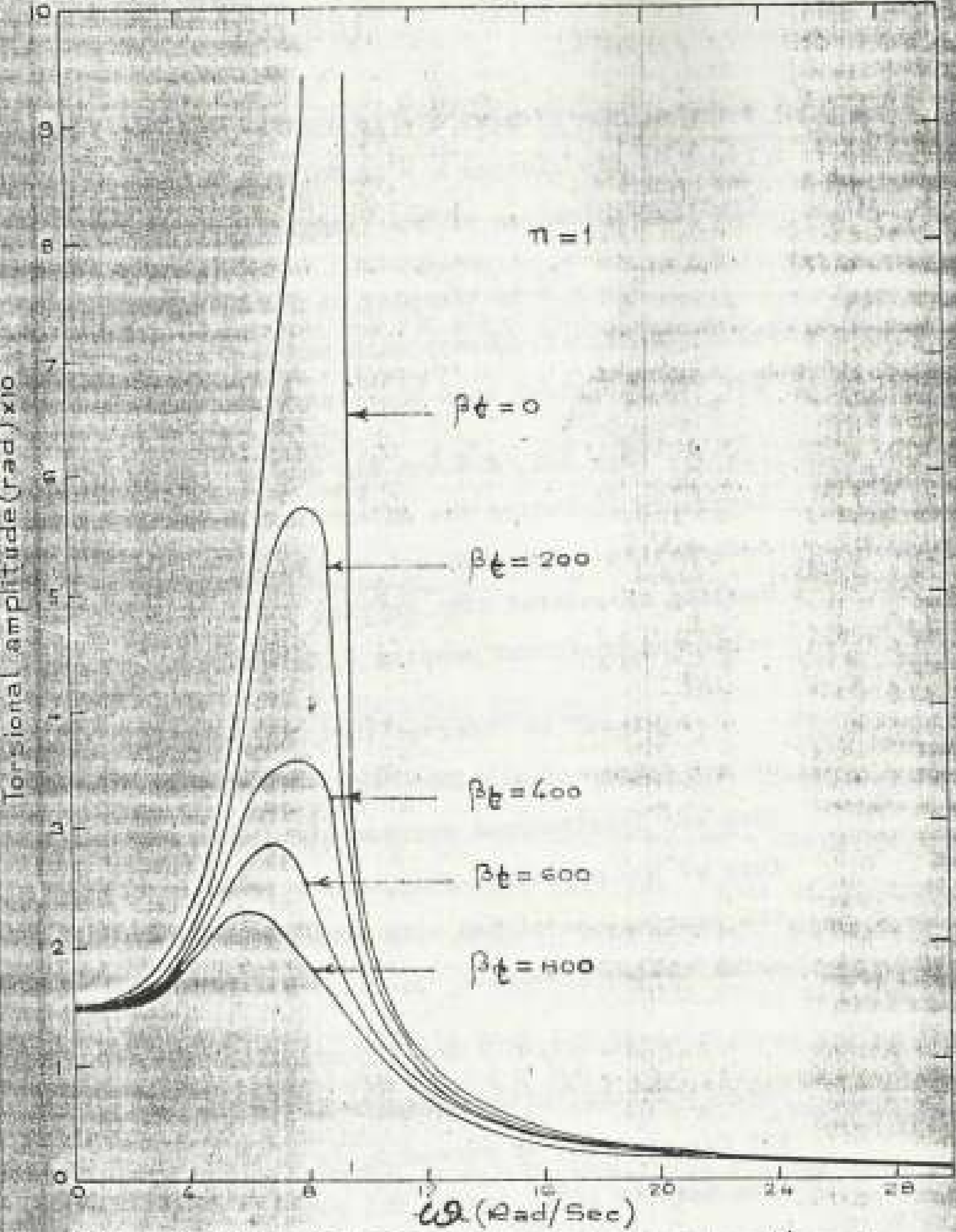


Fig.6.2. Classic beam Timoshenko torsion theory

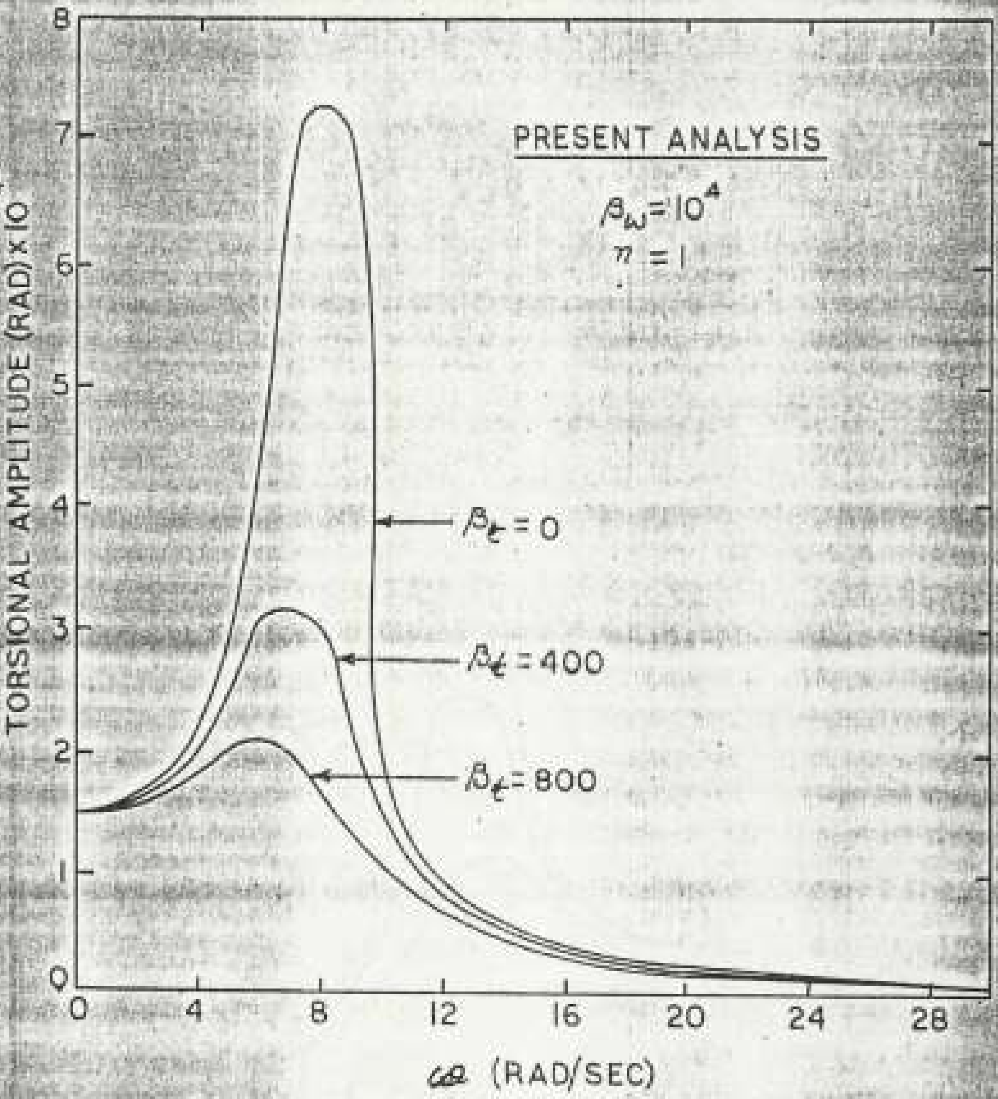


Fig. 6.3. Present analysis.

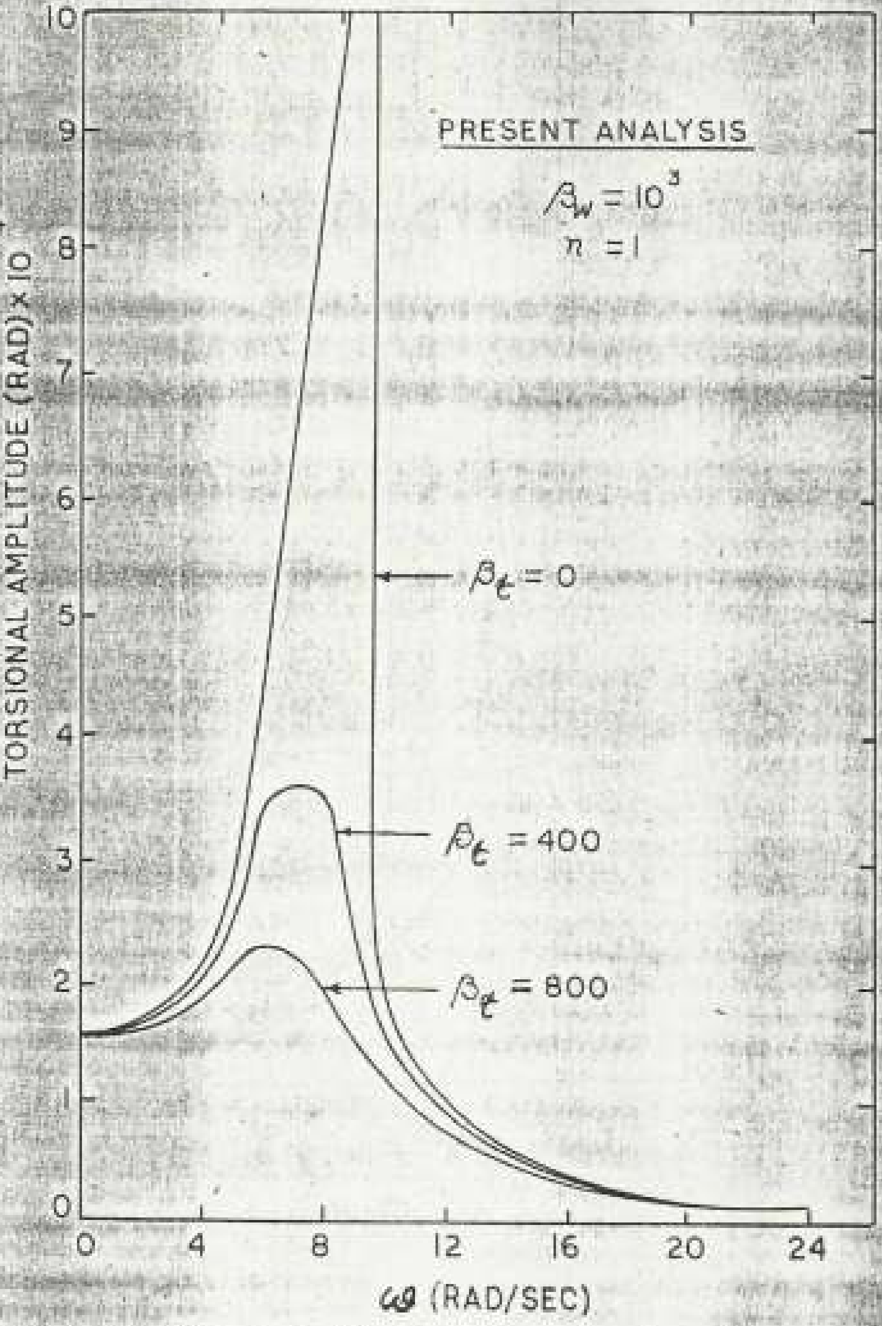


Fig. 6.4. Present analysis.

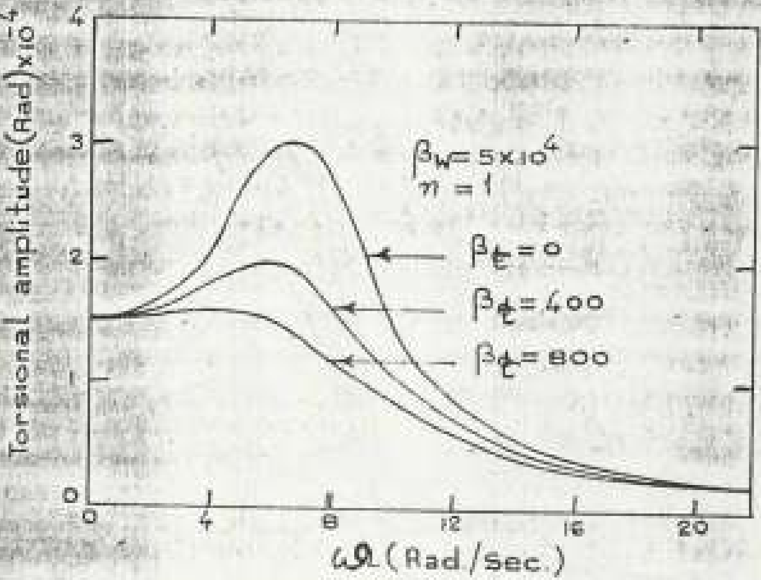


Fig. 6.5. Present analysis.

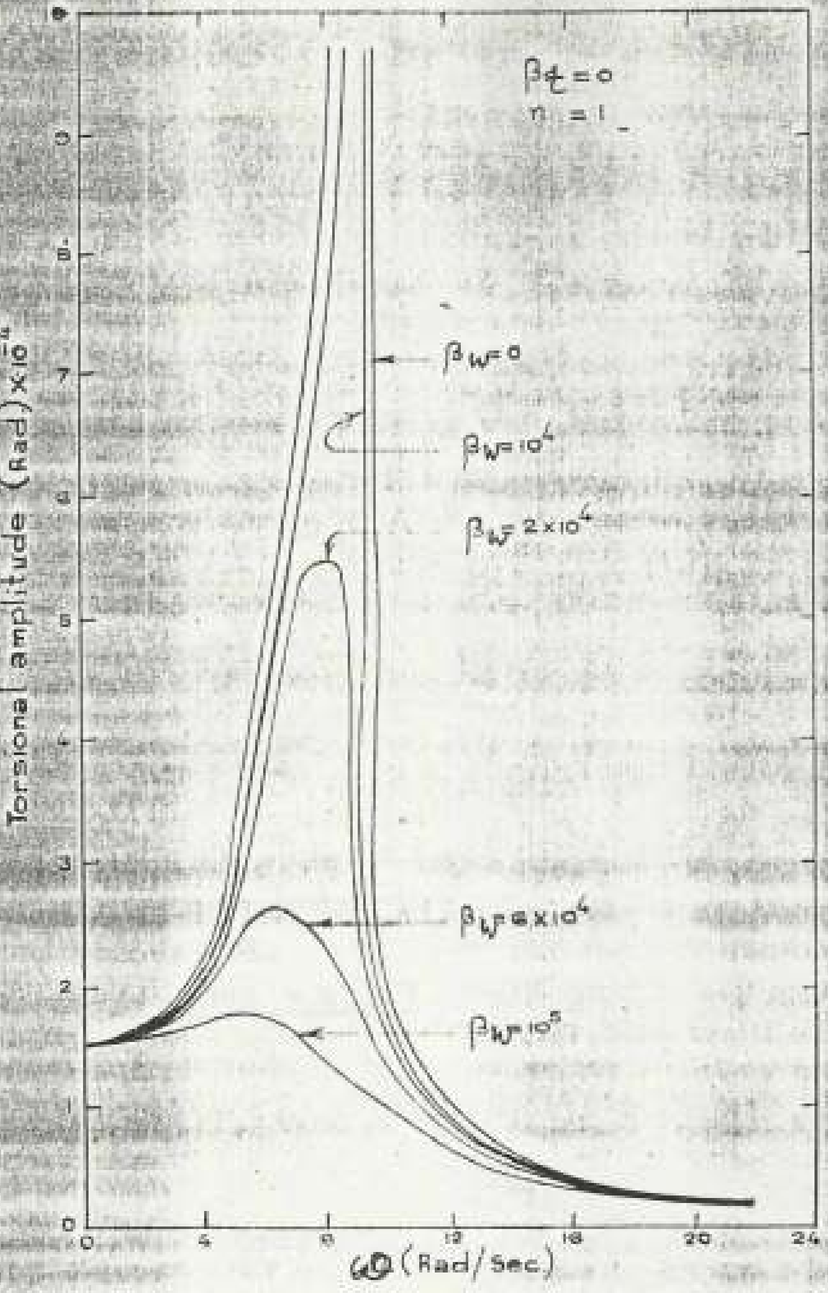


Fig. 6.6. Present analysis

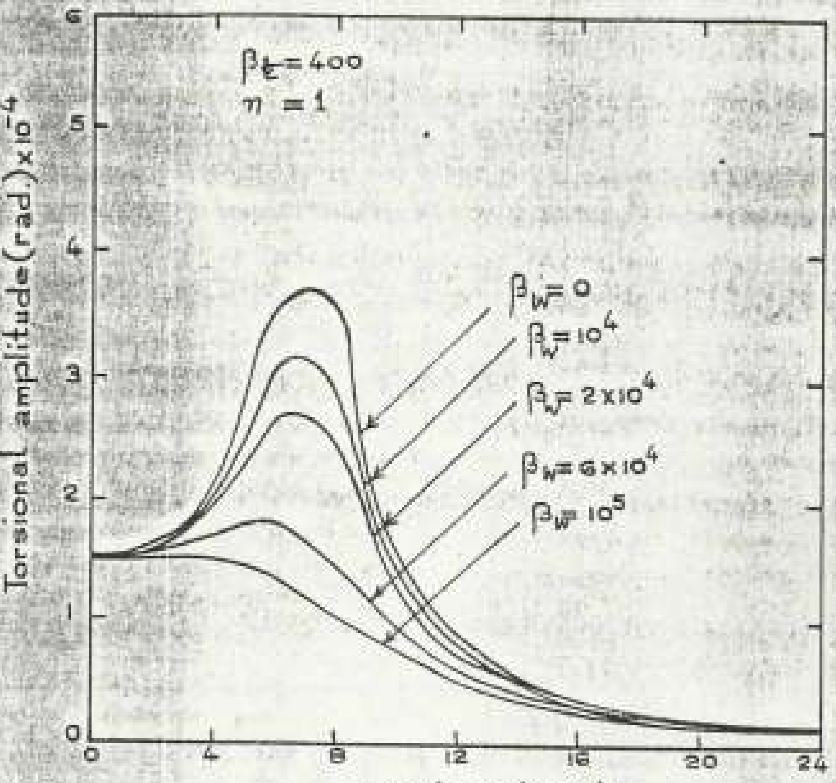


Fig. 6.7. Present Analysis.

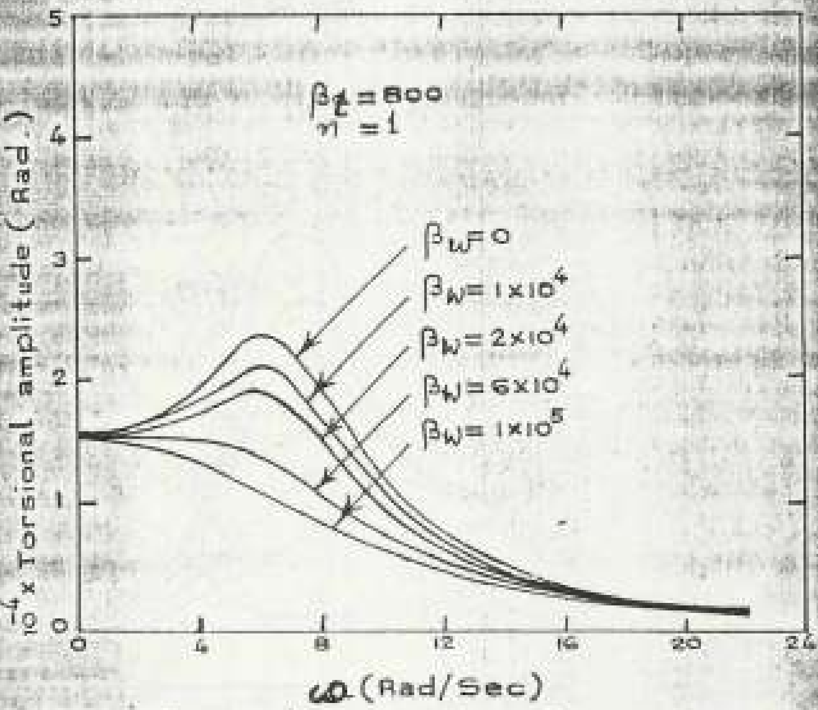


Fig. 6.8 Present analysis

TABLE - 6.1

Values of the natural frequencies and maximum total torsional amplitudes for various modes of vibration of a simply supported beam.

Mode Number n	Natural Frequency		Maximum Total Amplitude	
	Classic Beam	Present Analysis	Classic Beam	Present Analysis
1	245.211	235.791	1.38790×10^{-7}	1.47853×10^{-7}
3	2,171.970	1,662.560	5.89665×10^{-10}	9.47434×10^{-10}
5	6,025.440	3,558.770	4.59715×10^{-11}	12.36510×10^{-11}
7	11,805.600	5,539.010	8.55382×10^{-12}	36.90330×10^{-12}
9	19,512.500	7,515.080	2.43537×10^{-12}	15.78190×10^{-12}

of damping associated with warping angle β_w. Therefore, including the effects of longitudinal inertia and shear deformation, the torsional velocity damping is more significant than the warping-velocity damping.

Further, to consider the effects on higher modes, light torsional damping, (β_t=200, β_w=0) will be applied to a beam of large depth to length ratio. Keeping the same physical parameters as above, except letting L = 100 (in) to emphasize the shear deformation effects, the 'maximum total torsional amplitude' response may be computed. This is the maximum torsional amplitude obtained due to superposition of the responses of all modes when the separate natural frequencies are successively ex cited. Maximum total torsional amplitudes are given in Table 6.1, for the first nine symmetric mode shapes of the simply supported beam. From Table 6.1, it is observed that as the mode number n increases the difference between the natural frequencies of the classical beam and, those obtained from the present analysis including the effects of longitudinal inertia and shear deformation, also increases. As shown in Chapters IV and V, the natural frequencies obtained by including the effects of longitudinal inertia and shear deformation are lower than those for the classic beam. However, the amplitudes obtained including longitudinal inertia and shear deformation are larger than those for the classic beam.

CHAPTER - VII

TORSIONAL WAVE PROPAGATION IN ORTHOTROPIC THIN-WALLED BEAMS OF OPEN SECTION INCLUDING THE EFFECTS OF LONGITUDINAL INERTIA AND SHEAR DEFORMATION*

7.1. INTRODUCTION:

In the previous Chapters, free and forced torsional vibrations of short thin-walled beams of open section including the effects of longitudinal inertia and shear deformation are analyzed both by exact and approximate methods. The present Chapter deals with the important problem of torsional wave propagation in orthotropic thin-walled beams of open section including the second order effects.

Though there exists a good amount of work on the analysis of flexural wave propagation, comparable torsional wave analysis was virtually neglected and very few papers on this topic have been published. The reason is the fact that Coulomb theory gives the same first-mode results as the exact theory. The available information is almost limited to the circular cylindrical bars. Thus, there exists a lack of satisfactory approximate and exact theories for torsional wave propagation in non-circular bars, especially those used in structural applications such as thin-walled beams of open section.

---

* A paper by the author based on the results of this Chapter is accepted for publication in the Journal of the Aeronautical Society of India. See Ref.(54).

An inadequacy of St.Venant's classical torsion theory for short wave lengths was hinted at by Love (76), who suggested a correction for the longitudinal inertia associated with torsional deflection. Vlasov (107) also introduced the effect of longitudinal inertia in his torsional analysis of thin-walled beams. However, both the elementary theory and Love's or Vlasov's approximation have the same defects as do their counterparts in longitudinal wave-propagation theory. The dynamic equation used by Gere (32) in his torsion analysis was essentially that previously derived by Timoshenko (98) and included the effect of warping of the cross section. These equations are found to lead to physically absurd results for short wavelengths. Aggarwal and Cranch (4) presented a strength of materials theory including the effects of warping of the cross section, longitudinal inertia and shear deformation. This theory was found to lead to theoretically satisfactory results for the first mode of transmission over a wavelength spectrum which included moderately short wavelengths, and that it agreed with previous approximations for large wavelengths. The group velocity for the second mode was found to increase monotonically from zero for the longest waves to the bar velocity for very short wavelengths. This was in agreement in form with the higher modes of the exact theory for circular cylindrical bars (89,23).

All the above work, and a host number of other investigations involving torsional wave propagation phenomena in thin-walled beams, concerns isotropic materials. Anisotropic materials have not been approached to the best of author's knowledge. As is well known, anisotropy of the material introduced considerable complications in the computational part of the solution.

The present Chapter therefore, aims at investigating the problem of torsional wave propagation in orthotropic thin-walled beams of open section including the effects of longitudinal inertia and shear deformation, from the strength of materials approach. This approach is attractive for its physical directness. More specifically, the interest is to find what values of the wave frequency result from the elementary theory established for the anisotropic analog of the isotropic thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. To this end, the equation of motion for free torsional vibrations of thin-walled beams of open section of orthotropic material including the second order effects is established, analogous to that for isotropic material. It is shown herein that, for some anisotropic materials, the corrections due to longitudinal inertia and shear deformation may be of one order of magnitude greater than the correction in the isotropic case. Graphs are also given for the phase velocity versus inverse wavelength for various aspect ratios of beams of different materials.

7.2. ANALYSIS AND EXAMPLES:

For definiteness and simplicity, let us take the material of the thin-walled open section beam to be orthotropic, with one axis of elastic symmetry, z-axis, directed along the axis of the beam.

As is well known the fundamental equation of elementary theory of flange-bending retains its validity for anisotropic materials of the most general type, provided the isotropic Young's modulus is replaced by the modulus E_{zz} for extention-compression along the axis of the bar.

In symbols,

$$M = E_{zz} I_f \frac{\partial \psi}{\partial z} \qquad (7.1)$$

analagous to the Eq.(4.4) for the isotropic beams.

Now, in the derivation, in strength of materials, of the formula for the maximum shear stress in flange-bending,

$$\tau_{zx}(\max) = -\frac{QS_0}{I_f t_w}, \qquad (7.2)$$

no specific elastic properties of the material besides certain, symmetric conditions, are postulated. This equation, therefore, is certainly valid (in the same sense of strength of materials) for the elastic symmetrices involved in the orthotropic thin-walled open section beam characterized earlier. For such a beam, with G_{zx} as the pertinent shear modulus,

$$\tau_{zx} = G_{zx}\,\varepsilon_{sh} \qquad (7.3)$$

so that

$$-Q = K' A_f G_{zx}\,\varepsilon_{sh} \qquad (7.4)$$

where ϵ_{sh} is the shear strain at the center of the flange, x=0, given by

$$\epsilon_{sh} = \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \qquad (7.5)$$

In Eq.(7.2) all others being previously defined, S_o stands for the statical moment with respect to neutral axis. In Eq.(7.4) K' is the shear coefficient which depends upon the shape of the cross section and is given by

$$K' = \frac{I_f t_w}{S_o A_f} . \qquad (7.6)$$

There is no difference between Eqs.(7.1) and (7.4) and the corresponding equations in the isotropic case i.e., Eqs.(4.4) and (4.7) of Chapter IV, except for the moduli E_{zz} and G_{zx} standing for E and G. One can therefore avoid all the transformation and proceed directly to derive the frequency equation.

Following the procedure in Chapter IV, the equations of motion can be now written for torsional vibrations of orthotropic thin-walled beams of open section as:

$$G_{zx} C_s \frac{\partial^2 \phi}{\partial z^2} + K' A_f G_{zx} h \left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z} \right) = \rho I_p \frac{\partial^2 \phi}{\partial t^2} \qquad (7.7)$$

and

$$K' A_f G_{zx} \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) + E_{zz} I_f \frac{\partial^2 \psi}{\partial z^2} = \rho I_f \frac{\partial^2 \psi}{\partial t^2} \qquad (7.8)$$

Eliminating ψ between Eqs.(7.7) and (7.8) a single equation in ϕ may be obtained as:

$$\left[\frac{E_{zz} I_f C_s}{K' A_f G_{zx}} + E_{zz} C_w\right] \frac{\partial^4 \phi}{\partial z^4} - \left[\frac{\rho E_{zz} I_p I_f}{K' A_f G_{zx}} + \frac{\rho C_s I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right] \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- G_{zx} C_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} + \frac{\rho^2 I_p I_f}{K' A_f G_{zx}} \frac{\partial^4 \phi}{\partial t^4} = 0 \qquad (7.9)$$

For a wave-form solution in long beams, consider a sinusoidal wave,

$$\phi \sim e^{i\delta_1 (z - C_p t)} \qquad (7.10)$$

propagating along the beam. In Eq.(7.10), δ_1 is the wave number $= 2\pi/\Lambda$, Λ being the wavelength, C_p the phase velocity for torsional waves, and t is the time.

Substituting ϕ from Eq.(7.10) into Eq.(7.9), the frequency equation for torsional waves is obtained as

$$\frac{\rho I_f}{K'} \left(\frac{C_p}{C_2}\right)^4 - \left[\frac{\rho I_f}{K'}\left(\frac{E_{zz}}{G_{zx}}\right) + \frac{\rho I_f}{I_p}\left(\frac{C_s}{K'} + \frac{A_f h^2}{2}\right) + \frac{\rho A_f}{\delta_1^2}\right] \left(\frac{C_p}{C_2}\right)^2$$

$$+ \left[\frac{\rho I_f}{I_p}\left(\frac{E_{zz}}{G_{zx}}\right) \left(\frac{C_s}{K'} + \frac{A_f h^2}{2}\right) + \frac{\rho A_f C_s}{I_p \delta_1^2}\right] = 0 \qquad (7.11)$$

where $C_2 = (G_{zx}/\rho)^{1/2}$ is the shear wave velocity. Eq.(7.11) determines the phase velocities of the torsional wave propagation in an orthotropic thin-walled open section beam.

Two cases of interest can be deduced from Eq.(7.11) as follows:

(1) Neglecting shear deformation, by letting $K' \to \infty$, the frequency Eq. (7.11) reduces to:

$$\left(\frac{C_p}{C_2}\right)^2 = \frac{C_s + 2\pi^2 (E_{zz}/G_{zx})\, I_f (h/\Lambda)^2}{I_p + 2\pi^2\, I_f (h/\Lambda)^2} \qquad (7.12)$$

Eq. (7.12) therefore is the frequency equation which includes the warping and longitudinal inertia effects of the cross section.

(2) Neglecting longitudinal inertia and shear deformation, by letting $\rho I_f = 0$, $K' \to \infty$, the frequency equation (7.11) reduces to:

$$\left(\frac{C_p}{C_2}\right)^2 = \frac{1}{I_p}\left[C_s + 2\pi^2\, I_f (E_{zz}/G_{zx}) (h/\Lambda)^2 \right] \qquad (7.13)$$

which is the frequency equation including the effect of warping only and represents the Timoshenko tortion theory (32).

Returning now to the general Eq. (7.11) which includes both the second order effects, it may written in an alternative form as:

$$\left(\frac{C_p}{C_2}\right)^4 - \left[\bar{\alpha}_3 + \bar{\beta}_3 + \frac{\bar{\eta}_{L5}}{4\pi^2} \left(\frac{\Lambda}{h}\right)^2 \right] \left(\frac{C_p}{C_2}\right)^2 + \left[\bar{\alpha}_3 \bar{\beta}_3 + \frac{\bar{\eta}_{L}\bar{\zeta}_{2}}{4\pi^2} \left(\frac{\Lambda}{h}\right)^2 \right] = 0 \qquad (7.14)$$

where

$$\bar{\alpha}_3 = E_{zz}/G_{zx} \qquad (7.15)$$

$$\bar{\beta}_3 = \frac{1}{I_p}\left[C_s + (1/2)\, K' A_f h^2 \right] \qquad (7.16)$$

$$\bar{\eta}_{\xi} = K' A_f h^2/I_f \qquad (7.17)$$

and

$$\bar{\zeta}_2 = C_s/I_p \qquad (7.18)$$

Eq.(7.14) gives rise to to two modes of wave transmission. The new mode can be explained to arise from the coupled interaction of the torsional deformation with the bending effects of shear deformation and longitudinal inertia. The phase velocities for the two modes are given by Eq.(7.14) as:

$$\left(\frac{C_p}{C_2}\right)^2 = \frac{1}{2}\left\{ \left[\bar{\alpha}_3 + \bar{\beta}_3 + \frac{\bar{\eta}_{\xi}}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2 \right| \right.$$
$$\left. \pm \left[\left[\bar{\alpha}_3 + \bar{\beta}_3 + \frac{\bar{\eta}_{\xi}}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2\right]^2 - 4\left[\bar{\alpha}_3\bar{\beta}_3 + \frac{\eta_{\xi}\bar{\zeta}_2}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2\right]\right]^{1/2}\right\} \qquad (7.19)$$

where the minus sign is taken for the first mode.

Eq.(7.19) defines the phase velocity as a function of the shape of the cross section. At very large wave lengths the results for the lower mode obtained from Eq.(7.19) will agree with those from previous theories. This is obvious because the deformation associated with long wave lengths is primarily that of rotation of the cross section with essentially no warping, no shear deformation and hence no dispersion. The improved theory due to Aggarwal and Cranch (4) displays finite wave velocity $C_2\sqrt{\bar{\beta}_3}$ for very short wavelengths as against the infinite wave velocities predicted by Timoshenko torsion theory and low wave velocities predicted by Saint-Venant torsion theory.

From Eq.(7.16) which defines β_3, it may be observed that for short wave lengths, the torsional stiffness effect is very small and the shear distortion of the flanges contributes more. The present analysis gives satisfactory results for wave lengths $\Lambda > t_w$ for the first mode and this coincides in the second mode with the form of the exact theory for circular cylindrical bars. The range of applicability of the first mode, $\Lambda > t_w$, gives a wave length spectrum which includes moderately short waves and high frequencies, and as such covers a range of practical interest. As an example, for the beam for which $b/h = 0.75$, $t_f/h = 0.050$ and $t_w/h = 0.040$ the theory is valid for wave lengths $h/\Lambda < 25$.

Despite the fact that Eq.(7.19) has a form identical with that given by Aggarwal and Cranch (4) for isotropic beams, there is a basic difference between the two equations. It consists in that, for isotropic bodies, the value of poisson's ratio ranges (at least in principle) from 0 to 0.5, so that the value of E/G in Eq.(7.19) falls between 2 and 3. On the other hand for anisotropic materials the values of E_{zz}/G_{zx} may be one and possibly even two orders of magnitude higher. So much so, both the corrections due to shear deformation, and the corrections for longitudinal inertia and shear deformation together, may become several times greater for anisotropic beams than they are for isotropic beams.

Table 7.1. Values of $\bar{\alpha}_3$ for various materials.

Material	$\bar{\alpha}_3 = E_{zz}/G_{zx}$
Isotropy	2.6
Orthotropy II	13.9
Orthotropy I	17.1
Transverse Isotropy	35.0 (Average of the range 20 - 50)

The values of $\alpha_3 (= E_{zz}/G_{zx})$ for three types of anisotropic materials considered in this Chapter are given in Table 7.1. For an isotropic material the value of α is taken as 2.6.

7.3. RESULTS AND DISCUSSION:

Figs.7.1 to 7.8 show, the phase velocities for torsional waves in four wide-flanged I-beams which cover the practical range, having dimensions such as:

(1) $b/h=0.25$, $t_f/h=0.025$, $t_w/h=0.020$ (Figs.7.1 and 7.2)
(2) $b/h=0.50$, $t_f/h=0.040$, $t_w/h=0.025$ (Figs.7.3 and 7.4)
(3) $b/h=0.75$, $t_f/h=0.050$, $t_w/h=0.040$ (Figs.7.5 and 7.6)
(4) $b/h=1.00$, $t_f/h=0.10$, $t_w/h=0.050$ (Figs.7.7 and 7.8)

Of isotropic and three types of anisotropic materials having values of $\bar{\alpha}_3$, 2.6 (isotropic), 13.9 (orthotropy II), 17.1 (orthotropy I) and 35.0 (transverse isotropy). Figs.7.1, 7.3, 7.5 and 7.7 gives the results corresponding to the first mode for various values of

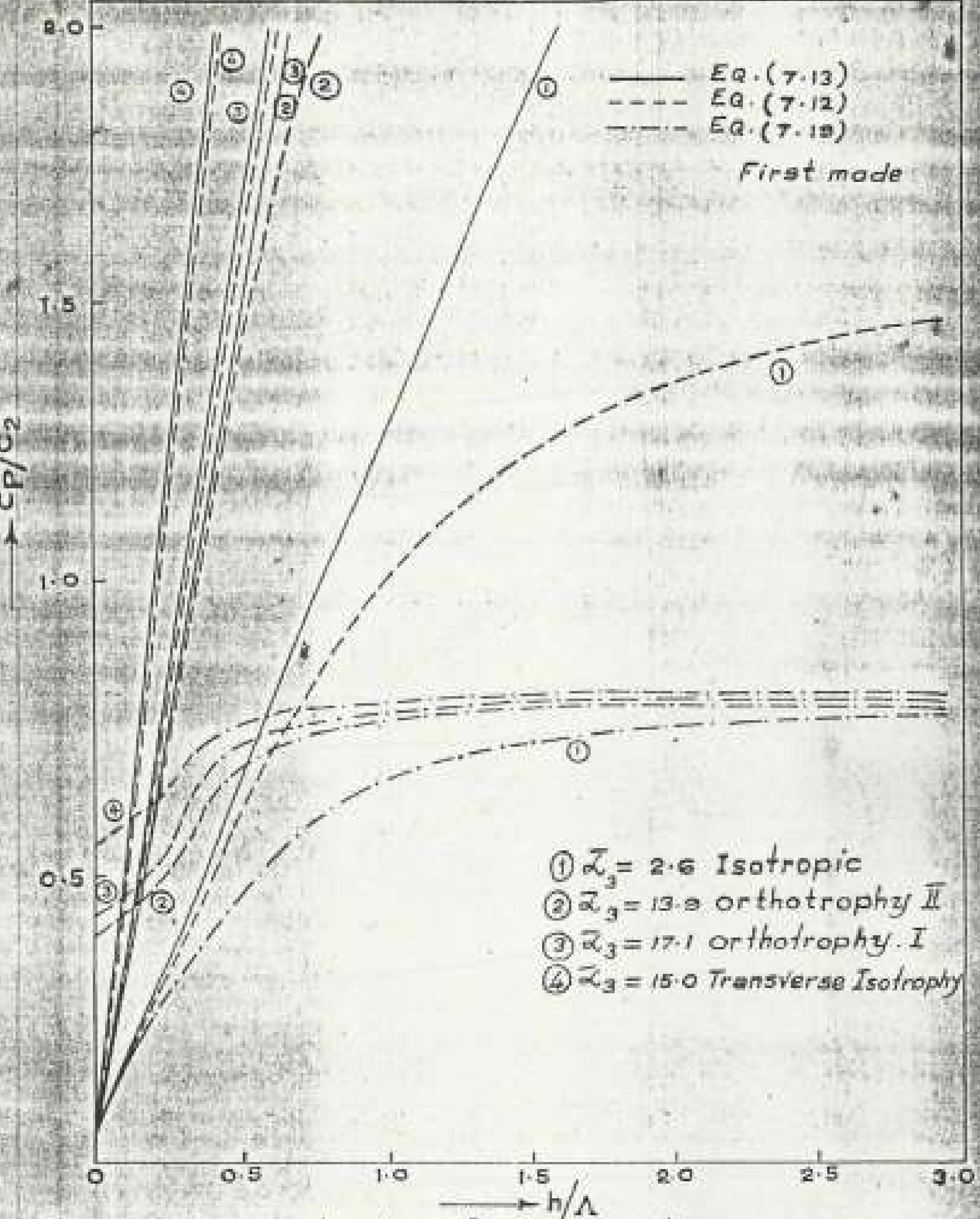


Fig. 7.1. phase velocities for torsional waves in I-beams

[$b/h = 0.50$; $t_f/h = 0.040$; $t_w/h = 0.025$]

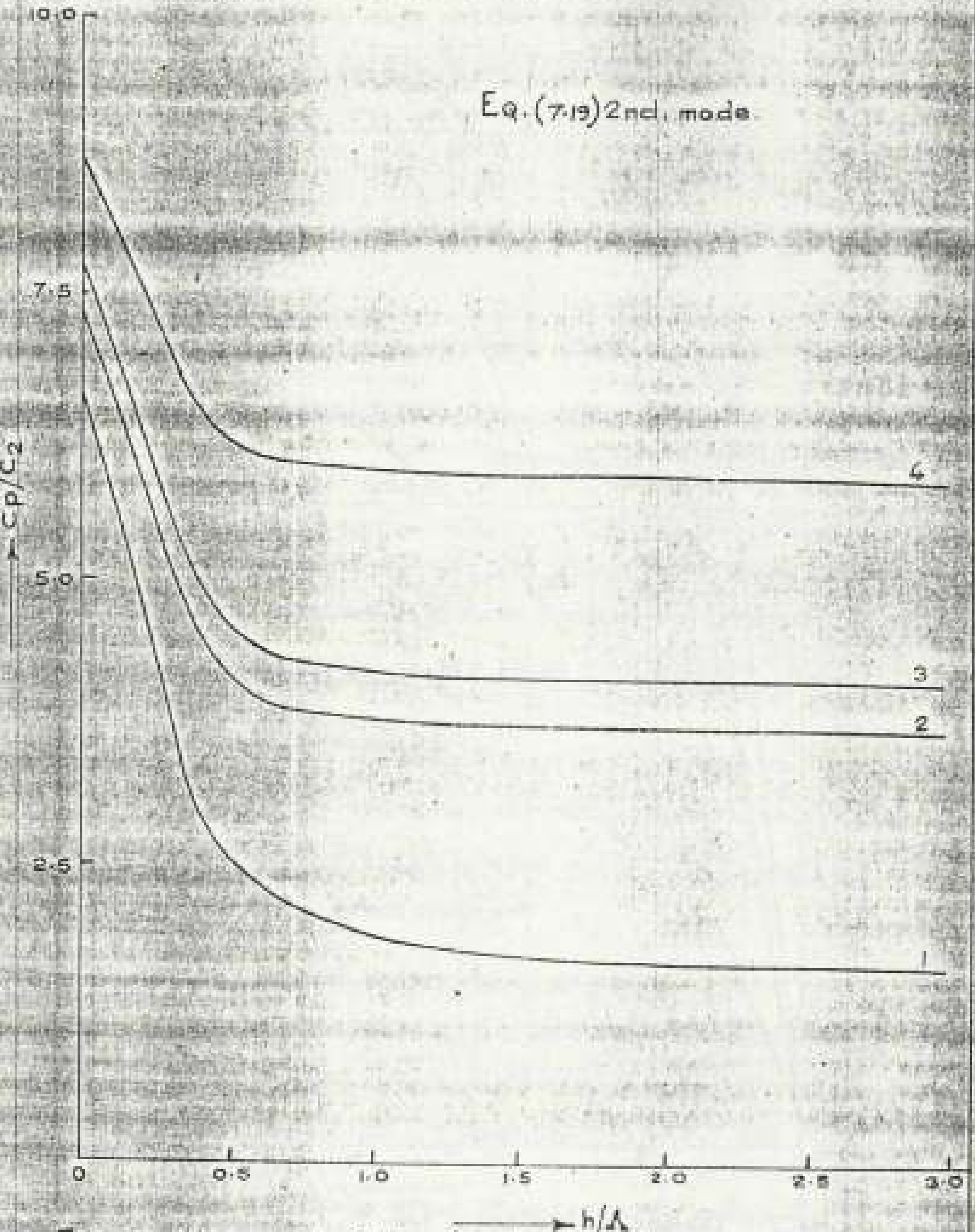


Fig. 7.2. phase velocities for torsional wave in I - beams.
[$b/h = 0.50$; $t_f/h = 0.040$; $t_w/h = 0.025$]

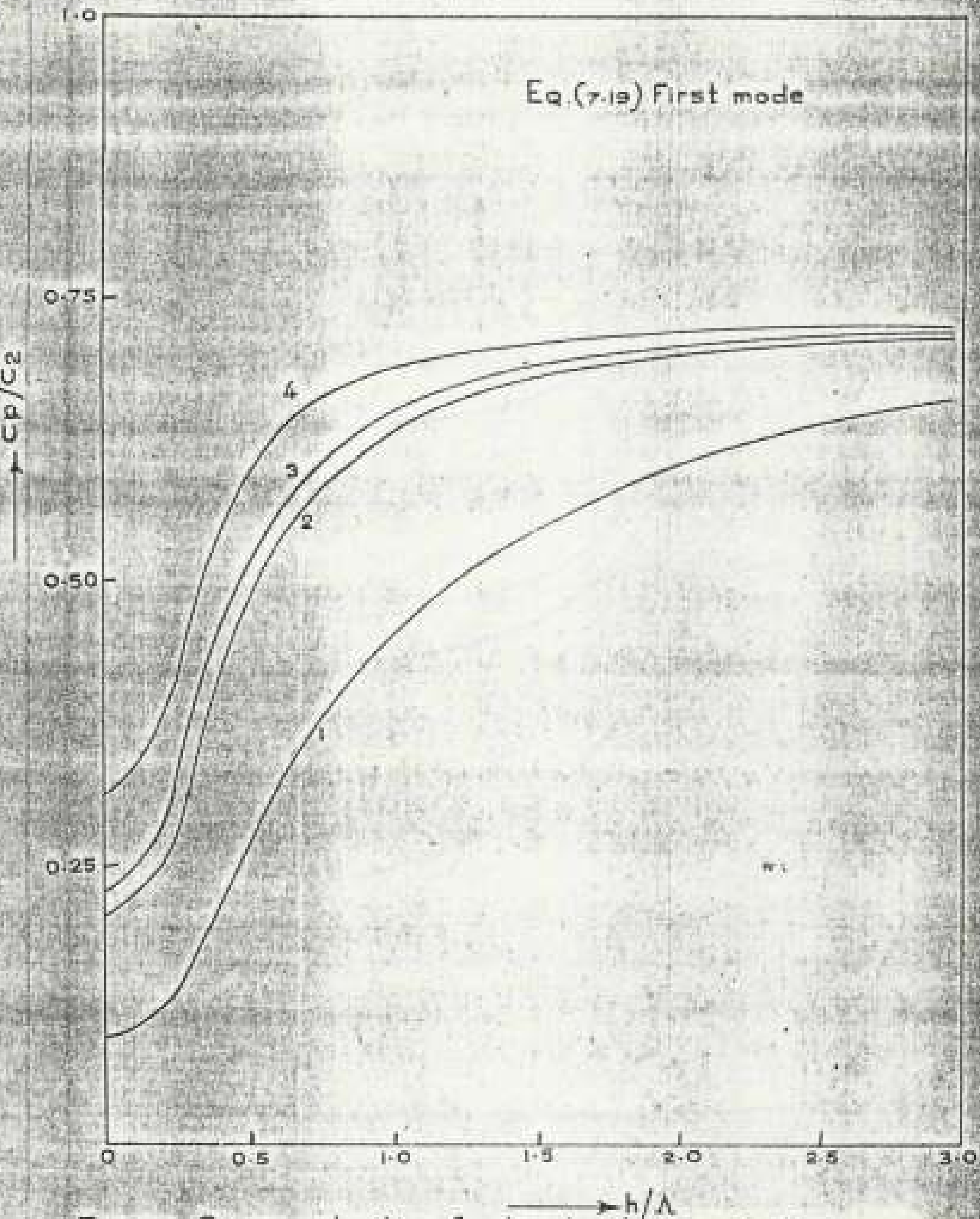


Fig. 7·3. Phase velocities for torsional waves in I-beams.
$[b_f/h = 0{\cdot}25;\ t_f/h = 0{\cdot}025;\ t_w/h = 0{\cdot}020]$

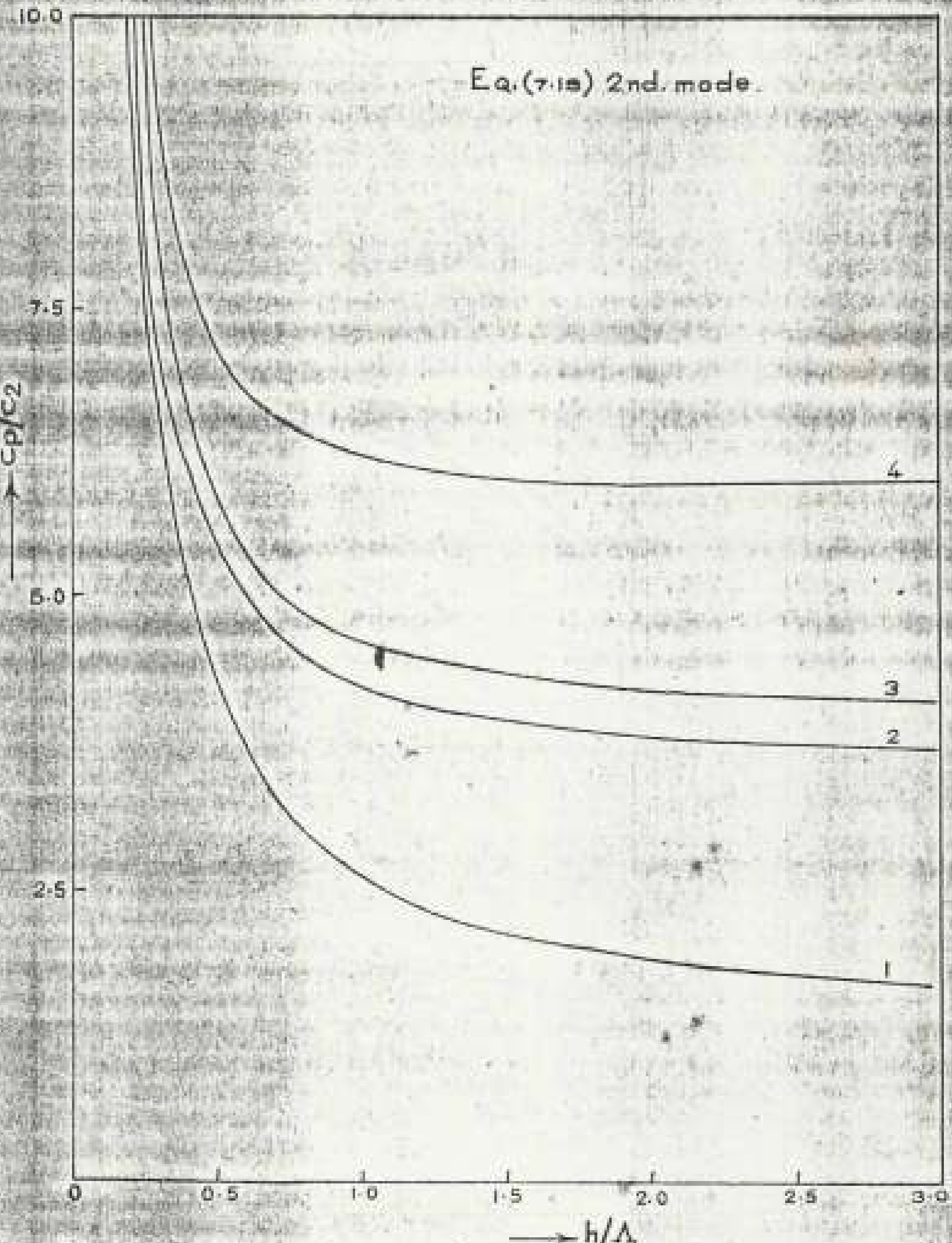


Fig. 7.4. Phase velocities for torsional waves in I-beams [$b_f/h = 0.25$; $t_f/h = 0.025$; $t_w/h = 0.020$]

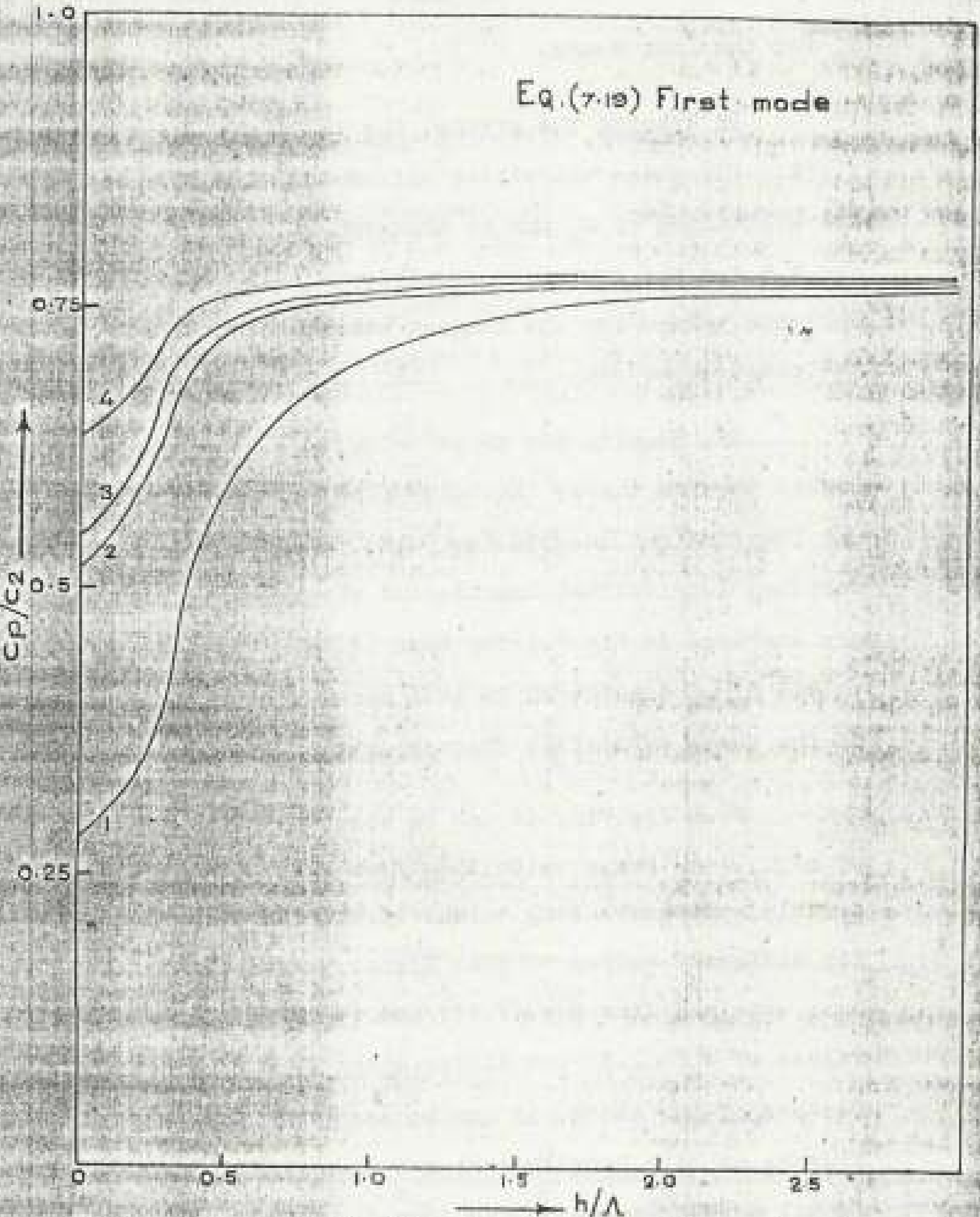


Fig. 7.5. Phase velocities for torsional waves in I-beams

[$b/h = 0.75$; $t_f/h = 0.050$; $t_w/h = 0.040$]

α_3 for the four beams.

In drawing the graphs, the value of K' was taken as $\pi^2/12$. The phase velocities corresponding to the second mode for all values of α_3 can be observed, from Figs.7.2, 7.4, 7.6 and 7.8 for the four beams considered here, to decrease from infinite values for the longest waves to the beam velocity for the shortest waves.

The results for phase velocities obtained from Timoshenko torsion theory (Eq.7.13), the theory including warping and longitudinal inertia (Eq.7.12), and the theory including warping, longitudinal inertia and shear deformation (Eq.7.19) are compared in Fig.7.1 for beam (1) defined above, for the four values of $\bar{\alpha}_3$ considered in this work. In all cases the values of the phase velocities increase with increasing values of $\bar{\alpha}_3$.

From Fig.7.1, it can be observed that, at lower values of h/Λ, the phase velocities from Eq.(7.19), increase considerably with increasing values of $\bar{\alpha}_3$, but differ only slightly for different values of α at higher values of h/Λ. The values obtained from Eqs.(7.12) and (7.13) differ greatly at lower values of $\bar{\alpha}_3$(= 2.6) but differ slightly for higher values of $\bar{\alpha}_3$. Because of the above, it can be seen, that the percentage of influence of both longitudinal and shear deformation on the torsional wave propagation may increase drastically for increasing values of $\bar{\alpha}_3$ i.e., E_{zz}/G_{zx}.

For example, for beam (1), for h/Λ = 0.4 and $\bar{\alpha}_3$= 2.6 (isotropic) the percentage influence of both longitudinal inertia

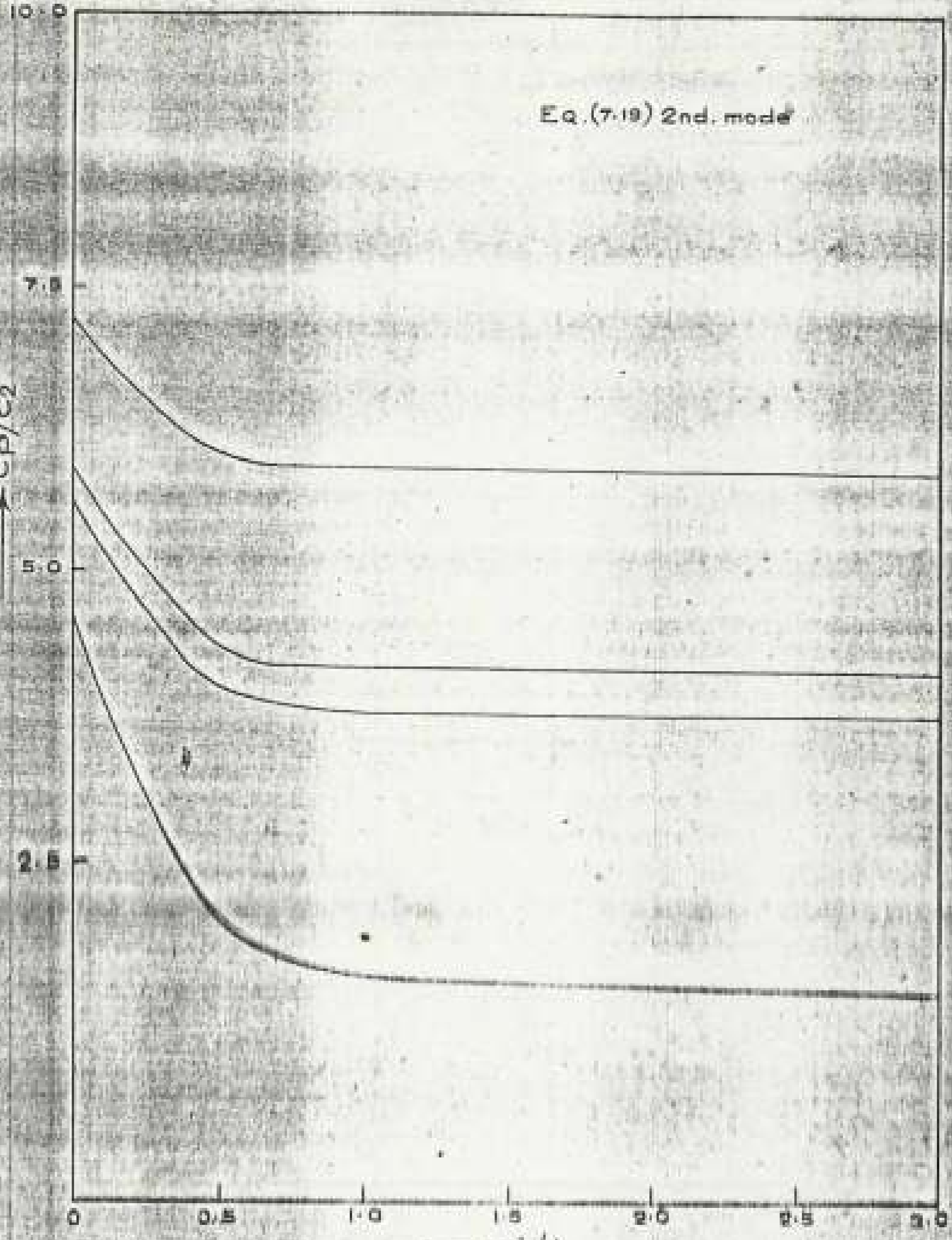


Fig. 7.6. Phase velocities for torsional waves in I-beams.
[$b/h = 0.75$; $t_f/h = 0.080$; $t_w/h = 0.040$]

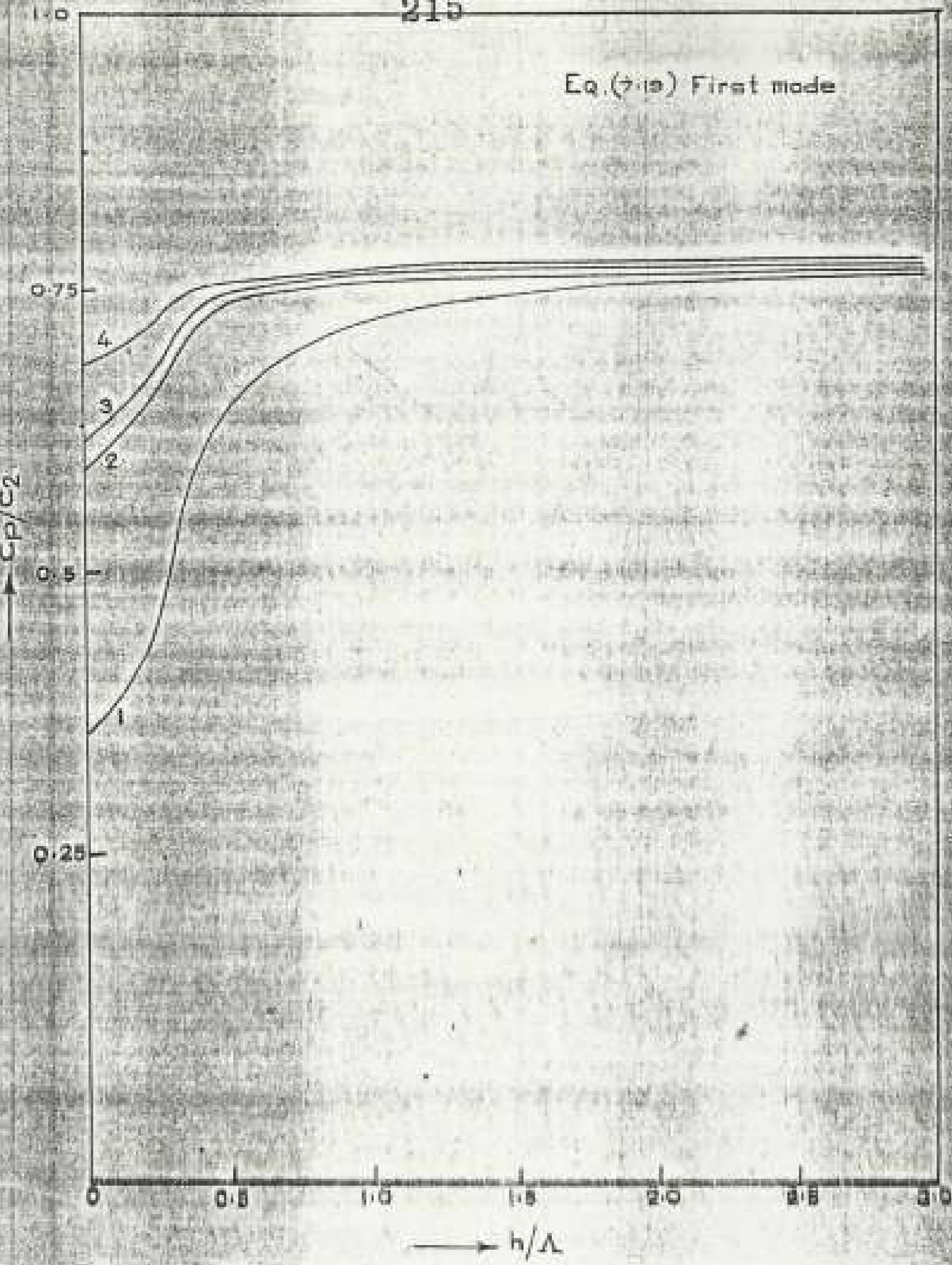


Fig.7.7. Phase velocities for torsional waves in I-beams.
[b/h = 1.00; t_f/h = 0.10; t_w/h = 0.050]

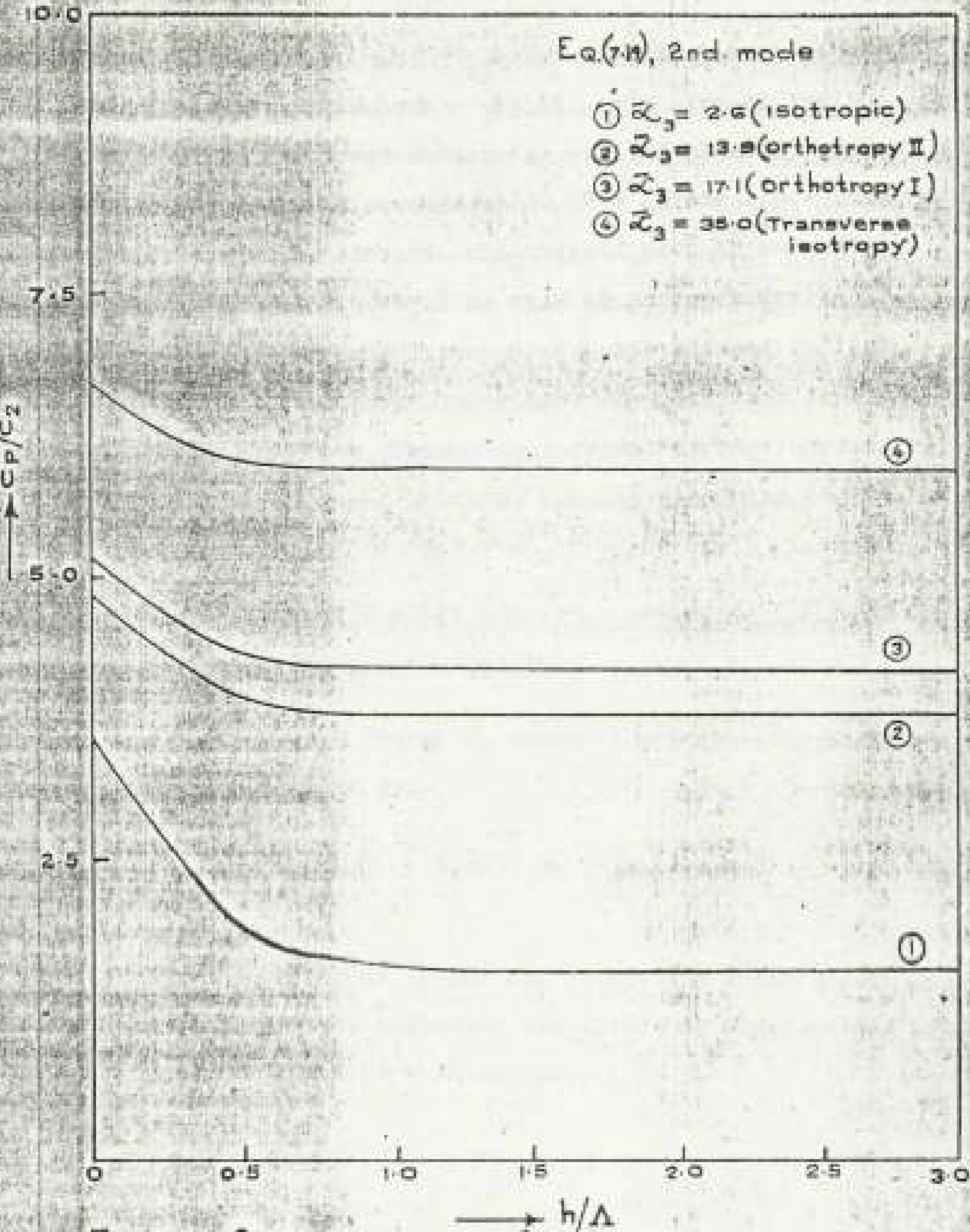


Fig. 7.8. Phase velocities for torsional waves in I beams. ($b/h = 1.00$, $t_f/h = 0.010$, $t_w/h = 0.080$)

and shear deformation is, $\delta_{ls} \approx 18$ percent and, that of longitudinal inertia alone is, $\delta_l \approx 4$ percent. But these values change drastically for anisotropic member and, for instance, for $h/\Lambda = 0.4$ and $\bar{\alpha}_3 = 35.0$ (transverse isotropy), the percentage influence of both longitudinal inertia and shear deformation for the first mode, is as high as $\delta_{ls} \approx 61$ percent and that of longitudinal inertia alone is $\delta_l \approx 4.7$ percent. Hence, it can be concluded that for some anisotropic materials, the corrections due to longitudinal inertia and shear deformation may be of one order of magnitude greater than the corrections in the isotropic case.

elastic foundation including the effects of longitudinal inertia and shear deformation. The coupled differential equations in angle of twist and warping angle governing the motion of the short thin-walled beam in torsion are derived utilizing Hamilton's principle. New frequency and normal mode equations which include the effects of time-invarient axial compressive load and elastic foundation are derived for various simple end conditions. The effects of axial load and elastic foundation, in combination with the second order influences, on the torsional frequencies and buckling loads are discussed for the case of a simply supported beam.

8.2. DERIVATION OF COUPLED EQUATIONS OF MOTION INCLUDING AXIAL LOAD AND ELASTIC FOUNDATION:

The strain energy U_4 ~~due to~~ in the Winkler-type elastic foundation is given by:

$$U_4 = \frac{1}{2} \int_0^L K_t (\phi)^2 \, dz \qquad (8.1)$$

Utilizing Eqs.(4.12) and (8.1), the total strain energy U at any instant t, including the effect of Winkler-type elastic foundation can be written as:

$$U = U_1 + U_2 + U_3 + U_4$$

$$= \frac{1}{2} \int_0^L \left[GC_s \left(\frac{\partial \phi}{\partial z}\right)^2 + 2\, EI_f \left(\frac{\partial \psi}{\partial z}\right)^2 + 2\, K' A_f G \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right)^2 + K_t (\phi)^2 \right] dz \qquad (8.2)$$

The potential energy, W, due to the time-invariant axial compressive load P is given by:

$$W = \frac{1}{2} \int_0^L \frac{PI_p}{A} \left(\frac{\partial \phi}{\partial z} \right)^2 dz \qquad (8.3)$$

The total kinetic energy at time t is

$$T_K = \frac{1}{2} \int_0^L \left[\rho I_p \left(\frac{\partial \phi}{\partial t}\right)^2 + 2 \rho I_f \left(\frac{\partial \psi}{\partial t}\right)^2 \right] dz \qquad (8.4)$$

which is same as Eq.(4.13).

If T_K, U and W from Eqs.(8.4), (8.2) and (8.3) are substituted into Eq.(2.1), and variations taken, and after integrating the first two terms by parts with respect to t and next five terms with respect to z, we obtain:

$$\int_{t_0}^{t_1} \int_0^L \left[\left\{ (GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + K' A_f Gh (\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z}) \right. \right.$$

$$\left. - K_t \phi - \rho I_p \frac{\partial^2 \phi}{\partial z} \right\} \bar{\delta}\phi + \left\{ 2 EI_f \frac{\partial^2}{\partial z^2} - 2 \rho I_f \frac{\partial^2 \psi}{\partial t^2} \right.$$

$$\left. \left. + 2 K' A_f G (\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) \right\} \bar{\delta}\psi \right] dz\, dt$$

$$+ \int_0^L (\rho I_p \frac{\partial \phi}{\partial t} \delta\phi + 2 \rho I_f \frac{\partial \psi}{\partial t} \delta \psi) \Big|_{t_0}^{t_1} dz$$

$$- \int_{t_0}^{t_1} \left[\left\{ (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} + K' A_f Gh (\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) \right\} \bar{\delta}\phi \right.$$

$$\left. + 2 EI_f \frac{\partial \psi}{\partial z} \bar{\delta}\psi \right|_0^L dt = 0 \qquad (8.5)$$

Assuming that the values of ϕ and ψ are given at the two fixed instants, the second integral vanishes. If the boundary conditions are such that the third integral also vanishes, then we obtain the two coupled equations of motion as:

$$(GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + K' A_f G h (\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z}) - K_t \phi - \rho I_p \frac{\partial^2 \phi}{\partial z^2} = 0$$

and (8.6)

$$EI_f \frac{\partial^2 \psi}{\partial z^2} + K' A_f G (\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) - \rho I_f \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad (8.7)$$

8.3. NATURAL BOUNDARY CONDITIONS:

In deriving the coupled equations (8.6) and (8.7) from (8.5) it was assumed that the expression

$$\left[(GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} + K' A_f G h (\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) \right] \delta\phi + 2\, EI_f \frac{\partial \psi}{\partial z} \; \delta\psi$$

vanishes at the ends $z=0$ and $z=L$. This condition is satisfied if at the two ends,

$$\left[(GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} + K' A_f G h (\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) \right] \delta\phi = 0 \qquad (8.8)$$

and

$$\frac{\partial \psi}{\partial z} \delta \psi = 0 \qquad (8.9)$$

Eqs.(8.8) and (8.9) give the natural boundary conditions for the finite bar. Except for the case of a free end, the boundary conditions for simply supported and fixed ends remain the same as those given by Eqs.(4.19) and (4.20).

For the case of a ''free end'', the natural boundary conditions for the present problem become:

$$\frac{\partial \psi}{\partial z} = 0, \text{ and } \left(GC_s - \frac{PI_p}{A}\right) \frac{\partial \phi}{\partial z} + K' A_f G h \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi\right) = 0 \qquad (8.10)$$

It can be observed that the difference between Eqs.(8.10) and (4.21) for the case of the free end is due to the presence of the axial compressive load, P, acting at the shear center (or centroid) of the beam.

8.4.1. SINGLE EQUATION IN ANGLE OF TWIST:

Eliminating ψ between the coupled Equations (8.6) and (8.7), a single equation of motion in angle of twist ϕ may be obtained as:

$$\left| \frac{EI_f C_s}{K' A_f} + EC_w - \frac{PI_p EI_f}{K' A_f G A} \right| \frac{\partial^4 \phi}{\partial z^4}$$

$$- \left| \frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2} - \frac{PI_p \rho I_f}{K' A_f G A} \right| \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- \left(GC_s + \frac{EI_f K_t}{K' A_f G} - \frac{PI_p}{A}\right) \frac{\partial^2 \phi}{\partial z^2} + \left(\rho I_p + \frac{\rho I_f K_t}{K' A_f G}\right) \frac{\partial^2 \phi}{\partial t^2}$$

$$+ \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial t^4} + K_t \phi = 0 \qquad (8,11)$$

Eq.(8.11) is the linear partial differential equation of fourth order governing the torsional vibrations and stability of a thin-walled beam resting on continuous elastic foundation.

8.4.1. ANALYSIS OF VARIOUS TERMS:

(i) Letting $C_w = \rho I_f = 0$ and $K' = \infty$, Eq.(8.11) reduces to:

$$(GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} - \rho I_p \frac{\partial^2\phi}{\partial t^2} - K_t\phi = 0 \qquad (8.12)$$

Eq.(8.12) represents the governing differential equation of motion for the torsional vibrations and stability of a beam resting on continuous elastic foundation, based on Saint Venant torsion theory and does not include~~d~~ the effects of warping, longitudinal inertia and shear deformation.

(ii) If $C_w = 0$ and $K' \to \infty$, then Eq.(8.11) becomes:

$$(GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} + \frac{\rho I_f h^2}{2}\frac{\partial^4\phi}{\partial z^2 \partial t^2} - \rho I_p \frac{\partial^2\phi}{\partial t^2} - K_t\phi = 0 \qquad (8.13)$$

Eq.(8.13) represents the equation of motion based on Love's torsion theory and includes the effect of longitudinal inertia.

(iii) If $\rho I_f = 0$ and $K' \to \infty$, Eq.(8.11) reduces to:

$$EC_w \frac{\partial^4\phi}{\partial z^4} - (GC_s - \frac{PI_p}{A})\frac{\partial^2\phi}{\partial z^2} + K_t\phi + \rho I_p \frac{\partial^2\phi}{\partial t^2} = 0 \qquad (8.14)$$

Eq.(8.14) is the governing differential equation of motion based on Timoshenko torsion theory which includes the effect of warping and neglects longitudinal inertia and shear deformation. It must be recalled that this equation is same as

Eq.(2.6) which is completely solved in Chapter II for various end conditions of the beam.

(iv) If $K' \rightarrow \infty$, Eq.(8.11) becomes:

$$EC_w \frac{\partial^4\phi}{\partial z^4} - \frac{\rho I_f h^2}{2}\frac{\partial^4\phi}{\partial z^2\partial t^2} - \left(GC_s - \frac{PI_p}{A}\right)\frac{\partial^2\phi}{\partial z^2} + K_t\phi + \rho I_p\frac{\partial^2\phi}{\partial t^2} = 0 \tag{8.15}$$

Eq.(8.15) represents the governing differential equation of motion including the effects of warping and longitudinal inertia but neglecting the effect of shear deformation.

(v) If $\rho I_f = 0$, Eq.(8.11) reduces to:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w - \frac{PI_p EI_f}{K' A_f G A}\right]\frac{\partial^4\phi}{\partial z^4} - \frac{E\rho I_p I_f}{K' A_f G}\frac{\partial^4\phi}{\partial z^2\partial t^2}$$

$$- \left(GC_s + \frac{EI_f K_t}{K' A_f G} - \frac{PI_p}{A}\right)\frac{\partial^2\phi}{\partial z^2} + \rho I_p\frac{\partial^2\phi}{\partial t^2} + K_t\phi = 0 \tag{8.16}$$

Eq.(8.16) is the equation of motion including the effects of warping and shear deformation but neglecting the effect of longitudinal inertia.

8.5. NON-DIMENSIONALIZATION AND GENERAL SOLUTION:

Eliminating ϕ in Eqs.(8.6) and (8.7) we obtain the complete differential equation in warping angle ψ as:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w - \frac{PI_p EI_f}{K' A_f GA}\right]\frac{\partial^4 \psi}{\partial z^4}$$

$$-\left[\frac{E\rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2} - \frac{PI_p \rho I_f}{K' A_f GA}\right]\frac{\partial^4 \psi}{\partial z^2 \partial t^2}$$

$$-\left(GC_s + \frac{EI_f K_t}{K' A_f G} - \frac{PI_p}{A}\right)\frac{\partial^2 \psi}{\partial z^2} + \left(\rho I_p + \frac{\rho I_f K_t}{K' A_f G}\right)\frac{\partial^2 \psi}{\partial t^2}$$

$$+\frac{\rho I_p \rho I_f}{K' A_f G}\frac{\partial^4 \psi}{\partial t^4} + K_t \psi = 0 \qquad (8.17)$$

Substituting Eqs.(4.30) to (4.32) and omitting the factor e^{ipt}, Eqs.(8.6), (8.7), (8.11) and (8.17) are reduced to:

$$\left[s^2(K^2-\Delta^2)+1\right]\bar{\phi}'' + s^2(\lambda^2-4\nu^2)\bar{\phi} - (2L/h)\bar{\psi}' = 0 \qquad (8.18)$$

$$s^2\bar{\psi}'' - (1-\lambda^2 s^2 d^2)\bar{\psi} + (h/2L)\bar{\phi}' = 0 \qquad (8.19)$$

$$\left[s^2(K^2-\Delta^2)+1\right]\bar{\phi}^{iv} + \left[\lambda^2 s^2 d^2 + \Delta^2(1-\lambda^2 s^2 d^2) + s^2(\lambda^2-4\nu^2)\right]\bar{\phi}''$$

$$-(\lambda^2-4\nu^2)(1-\lambda^2 s^2 d^2)\bar{\phi} = 0 \qquad (8.20)$$

$$\left[s^2(K^2-\Delta^2)+1\right]\bar{\psi}^{iv} + \left[\lambda^2 s^2 d^2 + \Delta^2(1-\lambda^2 s^2 d^2) + s^2(\lambda^2-4\nu^2)\right]\bar{\psi}''$$

$$-(\lambda^2-4\nu^2)(1-\lambda^2 s^2 d^2)\bar{\psi} = 0 \qquad (8.21)$$

where primes denote differentiation with respect to Z.

The general solutions of Eqs.(8.20) and (8.21) can be found as:

$$\bar{\phi} = B_1 \cosh \alpha_3 Z + B_2 \sinh \alpha_3 Z + B_3 \cos \beta_3 Z + B_4 \sin \beta_3 Z \tag{8.22}$$

$$\bar{\psi} = B_1' \sinh \alpha_3 Z + B_2' \cosh \alpha_3 Z + B_3' \sin \beta_3 Z + B_4' \cos \beta_3 Z \tag{8.23}$$

where

$$\begin{matrix}\alpha_3\\ \beta_3\end{matrix} = \frac{1}{\sqrt{2}\left[s^2(K^2-\triangle^2)+1\right]^{1/2}} \left\{ \mp \left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) + s^2(\lambda^2 - 4\gamma^2)\right] + \left[\left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) - s^2(\lambda^2 - 4\gamma^2)\right]^2 + 4(\lambda^2 - 4\gamma^2)\right]^{1/2} \right\}^{1/2} \tag{8.24}$$

and

$$\left\{\left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 s^2) - s^2(\lambda^2 - 4\gamma^2)\right]^2 + 4(\lambda^2 - 4\gamma^2)\right\}^{1/2} > \left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) + s^2(\lambda^2 - 4\gamma^2)\right]$$

is assumed.

In case

$$\left\{\left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) - s^2(\lambda^2 - 4\gamma^2)\right]^2 + 4(\lambda^2 - 4\gamma^2)\right\}^{1/2} < \left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) + s^2(\lambda^2 - 4\gamma^2)\right]$$

we write

$$\alpha_3 = \frac{1}{\sqrt{2}\left[s^2(K^2-\triangle^2)+1\right]^{1/2}} \left\{ \left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) + s^2(\lambda^2 - 4\gamma^2)\right] - \left[\left[\lambda^2 a^2 d^2 + \triangle^2(1-\lambda^2 s^2 d^2) - s^2(\lambda^2 - 4\gamma^2)\right]^2 + 4(\lambda^2 - 4\gamma^2)\right]^{1/2} \right\}^{1/2} = i\,\alpha_3' \tag{8.25}$$

Then Eqs.(8.22) and (8.23) are replaced by

$$\bar{\phi} = B_1 \cos \alpha_3' Z + iB_2 \sin \alpha_3' Z + B_3 \cos \beta_3 Z + B_4 \sin \beta_3 Z \qquad (8.26)$$

$$\bar{\psi} = iB_1' \sin \alpha_3' Z + B_2' \cos \alpha_3' Z + B_3' \sin \beta_3 Z + B_4' \cos \beta_3 Z \qquad (8.27)$$

Solutions of Eqs.(8.22) and (8.23) or (8.26) and (8.27) are naturally the solutions of the original coupled equations (8.6) and (8.7).

Only one half of the constants in Eqs.(8.22) and (8.23) are independent. They are related by Eqs.(8.6) and (8.7) as follows:

$$B_1 = \frac{2L}{h\alpha_3}\left[1 - s^2(\alpha_3^2 + \lambda^2 d^2)\right]B_1' \qquad (8.28)$$

$$B_2 = \frac{2L}{h\alpha_3}\left[1 - s^2(\alpha_3^2 + \lambda^2 d^2)\right]B_2' \qquad (8.29)$$

$$B_3 = -\frac{2L}{h\beta_3}\left[1 + s^2(\beta_3^2 - \lambda^2 d^2)\right]B_3' \qquad (8.30)$$

$$B_4 = \frac{2L}{h\beta_3}\left[1 + s^2(\beta_3^2 - \lambda^2 d^2)\right]B_4' \qquad (8.31)$$

or

$$B_1' = \frac{h}{2L\alpha_3}\left\{\alpha_3^2\left[s^2(K^2 - \Delta^2) + 1\right] + s^2(\lambda^2 - 4\gamma^2)\right\}B_1 \qquad (8.32)$$

$$B_2' = \frac{h}{2L\alpha_3}\left\{\alpha_3^2\left[s^2(K^2 - \Delta^2) + 1\right] + s^2(\lambda^2 - 4\gamma^2)\right\}B_2 \qquad (8.33)$$

$$B_3' = -\frac{h}{2L\beta_3}\left\{\beta_3^2\left[s^2(K^2 - \Delta^2) + 1\right] - s^2(\lambda^2 - 4\gamma^2)\right\}B_3 \qquad (8.34)$$

$$B_4' = \frac{h}{2L\,\beta_3}\quad \beta_3^2\left[s^2(K^2-\Delta^2)+1\right] - s^2(\lambda^2-4\gamma^2)\;B_4 \qquad (8.35)$$

8.6. FREQUENCY OR BUCKLING LOAD EQUATIONS AND MODAL FUNCTIONS:

In section 8.3, natural boundary conditions for the present problem are discussed. By combining these conditions in pairs, many types of single-span beams can be analyzed. In terms of non-dimensional parameters, the boundary conditions for a ''free end'' can be written as:

$$\bar{\psi}' = 0,\ \left[s^2(K^2-\Delta^2)+1\right]\bar{\phi}' - (2L/h)\bar{\psi} = 0 \qquad (8.36)$$

The application of appropriate boundary conditions (4.56), (4.57) and (8.36) and, relations of integration constants (8.28) to (8.35) to Eqs.(8.22) and (8.23) yields for each type of beam a set of four constants B_1 to B_4 with or without primes. In order that solutions other than zero may exist the determinant of the coefficients of B's must be equal to zero. This leads to the frequency equations in each case and the roots of these frequency or buckling load equations, λ_i, $i = 1,2,3,\ldots n$, or Δ^2_{cr}, give the eigen values of the problem. The corresponding modal functions, $\bar{\phi}_i$ and $\bar{\psi}_i$ can be obtained accordingly.

8.6.1. SIMPLY SUPPORTED BEAM:

The boundary conditions for a beam simply supported at both ends are:

$$\bar{\phi} = \bar{\psi}' = 0 \qquad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at} \quad Z = 1$$

For the boundary conditions at Z = 0, Eqs.(8.22) and (8.23) give:

$$B_1 + B_3 = 0 \tag{8.37}$$

$$\left\{\alpha_3^2\left[s^2(K^2-\triangle^2)+1\right]+s^2(\lambda^2-4\gamma^2)\right\} B_1$$

$$-\left\{\beta_3^2\left[s^2(K^2-\triangle^2)+1\right]-s^2(\lambda^2-4\gamma^2)\right\} B_3 = 0 \tag{8.38}$$

Since the secular determinant, ie.,

$$\left[s^2(K^2-\triangle^2)+1\right](\alpha_3^2+\beta_3^2) \neq 0,$$

therefore it follows that $B_1 = B_3 = 0$. (8.39)

For the second pair of conditions at Z = 1, Eqs.(8.22) and (8.23) give:

$$B_2 \sinh \alpha_3 + B_4 \sin \beta_3 = 0 \tag{8.40}$$

and

$$\left\{\alpha_3^2\left[s^2(K^2-\triangle^2)+1\right]+s^2(\lambda^2-4\gamma^2)\right\} B_2 \sinh \alpha_3$$

$$-\ \beta_3^2\left[s^2(K^2-\triangle^2)+1\right]-s^2(\lambda^2-4\gamma^2)\Big\} B_4 \sin \beta_3 = 0 \tag{8.41}$$

For a non-trivial solution, the secular determinant must vanish. This gives the characterestic equation:

$$\left[s^2(K^2-\triangle^2)+1\right](\alpha_3^2+\beta_3^2)\sinh \alpha_3 \sin \beta_3 = 0 \tag{8.42}$$

Since $\left[s^2(K^2-\triangle^2)+1\right](\alpha_3^2+\beta_3^2) \neq 0$

and

$$\alpha_3 \neq 0,$$

From Eq.(8.42) we have

$$\beta_3 = n\pi, \; n = 1,2,3,\ldots \qquad (8.4$$

which leads to the main solution of the problem.

Letting $\beta_3^2 = n^2\pi^2$ in Eq.(8.24), the frequency equation in λ^2 is obtained as:

$$s^2d^2\lambda^4 - \lambda^2 \left\{1 + n^2\pi^2\left[s^2+d^2+s^2d^2(K^2-\triangle^2)\right]+4s^2d^2\gamma^2\right\} + \left\{n^4\pi^4\left[s^2(K^2-\triangle^2)+1\right]+n^2\pi^2(K^2-\triangle^2)+4\gamma^2(1+n^2\pi^2s^2)\right\} = 0 \qquad (8.44$$

This equation gives two real positive roots:

$$\lambda^2_{mn} = \frac{1}{2s^2d^2}\left\{\left[1+n^2\pi^2\left\{s^2+d^2+s^2d^2(K^2-\triangle^2)\right\}+4s^2d^2\gamma^2\right] + (-1)^m\left\{\left[1+n^2\pi^2\left\{s^2-d^2-s^2d^2(K^2-\triangle^2)\right\}-4s^2d^2\gamma^2\right]^2+4n^2\pi^2d^2\right\} \qquad (8.4$$

This frequency equation (8.45) in λ^2, has an infinite number of roots which in general represent two coupled frequency spectra.

Using Eqs.(8.43), (8.40) and (8.41), one gets:

$$B_2 = 0 \qquad (8.4$$

The modal functions are obtained from Eqs.(4.22) and (4.23) with B's given by Eqs.(8.39) and (8.46). These are given as:

$$\bar{\phi}_{mn} = \sin n\pi Z \tag{8.47}$$

$$\bar{\psi}_{mn} = \frac{-h}{2n\pi L}\left\{n^2\pi^2\left[s^2(K^2-\Delta^2)+1\right]-s^2(\lambda^2_{mn}-4\gamma^2)\right\}\cos n\pi Z \tag{8.48}$$

where λ^2_{mn} being given by (8.45).

The second spectrum appears at higher frequencies, greater than the critical frequency λ_c given by

$$\lambda_c^2 = 1/s^2d^2$$

and is due to interaction between shear deformation and longitudinal inertia. It should be mentioned here that for the range of values of the dimensionless parameters covered in this chapter, λ is less than λ_c.

For the case, $\lambda > \lambda_c$, it is convenient to use $\alpha_3 = i\alpha_3'$ and, the characterestic frequency equation (8.42) transforms to:

$$\sin\alpha_3' \sin\beta_3 = 0 \tag{8.49}$$

Hence, in case there is any extension from there on for λ beyond λ_c ie., $\lambda^2s^2d^2>1$, care should be taken to account for the frequencies of the second spectrum which can be obtained from Eq.(8.49).

By putting $s^2= d^2= 0$, in Eq.(8.44), the equation for the the frequency parameter λ, neglecting the effects of shear deformation and longitudinal inertia, can be obtained as:

$$\lambda^2 = n^2\pi^2(n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2 \qquad (8.50)$$

which is the same as Eq.(2.47) derived in Chapter-II utilizing Timoshenko torsion theory.

8.6.2. FIXED-FIXED BEAM:

For a beam clamped at both ends, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 1.$$

Applying the above boundary conditions to the general solutions, Eqs.(8.22) and (8.23), the frequency equation, for the first set ($\lambda < \lambda_c$) can be obtained as:

$$2 - 2\cosh\alpha_3\cos\beta_3 + \frac{(1 - \delta_1^2\theta_1^2)}{\delta_1\theta_1}\sinh\alpha_3\sin\beta_3 = 0 \qquad (8.51)$$

where

$$\delta_1 = \alpha_3/\beta_3 \qquad (8.52)$$

and

$$\theta_1 = \frac{\beta_3^2\left|s^2(K^2 - \Lambda^2) + 1\right| - s^2(\lambda^2 - 4\gamma^2)}{\alpha_3^2\left|s^2(K^2 - \Lambda^2) + 1\right| + s^2(\lambda^2 - 4\gamma^2)} \qquad (8.53)$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$2-2\cos\alpha_3'\cos\beta_3+\frac{(1+\delta_2^2\theta_2^2)}{\delta_2\theta_2}\sin\alpha_3'\sin\beta_3 = 0 \tag{8.54}$$

where

$$\delta_2 = \alpha_3' / \beta_3 \tag{8.55}$$

and

$$\theta_2' = -\frac{\beta_3^2\left[s^2(K^2-\triangle^2)+1\right]-s^2(\lambda^2-4\gamma^2)}{\alpha_3^2\left|s^2(K^2-\triangle^2)+1\right|-s^2(\lambda^2-4\gamma^2)} \tag{8.56}$$

The modal functions for the first set are given by:

$$\bar{\phi} = D(\cosh\alpha_3Z+\delta_1\eta_1^*\theta_1\sinh\alpha_3Z-\cos\beta_3Z+\eta_1^*\sin\beta_3Z) \tag{8.57}$$

$$\bar{\psi} = H(\cosh\alpha_3Z+\frac{\mu_1^*}{\delta_1\theta_1}\sinh\alpha_3Z-\cos\beta_3Z+\mu_1^*\sin\beta_3Z) \tag{8.58}$$

where

$$\eta_1^* = \frac{-\cosh\alpha_3+\cos\beta_3}{\delta_1\theta_1\sinh\alpha_3-\sin\beta_3} \tag{8.59}$$

$$\mu_1^* = \frac{-\cosh\alpha_3+\cos\beta_3}{(1/\delta_1\theta_1)\sinh\alpha_3+\sin\beta_3} \tag{8.60}$$

The modal functions for the second set are:

$$\bar{\phi} = D(\cos\alpha_3'Z-\delta_2\eta_2^*\theta_2\sin\alpha_3'Z-\cos\beta_3Z+\eta_2^*\sin\beta_3Z) \tag{8.61}$$

$$\bar{\psi} = H(\cos\alpha_3'Z+\frac{\mu_2^*}{\delta_2\theta_2}\sin\alpha_3'Z-\cos\beta_3Z+\mu_2^*\sin\beta_3Z) \tag{8.62}$$

where

$$\eta_2^* = \frac{\cos\alpha_3'-\cos\beta_3}{\delta_2\theta_2\sin\alpha_3'+\sin\beta_3} \tag{8.63}$$

$$\mu_2^* = \frac{-\cosh \alpha_3' + \cos \beta_3}{(1/\delta_2\theta_2)\sin \alpha_3' + \sin \beta_3} \qquad (8.64)$$

Since the coefficients in $\bar{\phi}$ and $\bar{\psi}$ of Eqs.(8.22) and (8.23) are related, the coefficients D and H, that appear in the modal functions given above, are connected through any one of the Eqs.(8.28) to (8.31) or (8.32) to (8.35).

8.6.3. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

With the end Z = 0, taken as clamped end, and with the end Z = 1 as the simply supported end, the boundary conditions are:

$$\bar{\phi} = \bar{\gamma} = 0 \quad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\gamma} = 0 \quad \text{at} \quad Z = 1$$

The frequency equation obtained from applying the above boundary condition to the general solutions, Eqs.(8.22) and (8.23 for the first set ($\lambda < \lambda_c$) is given by:

$$\delta_1\theta_1 \tanh \alpha_3 - \tan \beta_3 = 0 \qquad (8.65)$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$\delta_2\theta_2 \tan \alpha_3' + \tan \beta_3 = 0 \qquad (8.66)$$

The modal functions for the first set are given by:

$$\bar{\phi} = D(\text{Cosh } \alpha_3 Z - \coth \alpha_3 \sinh \alpha_3 Z - \cos \beta_3 Z + \cot \beta_3 \sin \beta_3 Z) \qquad (8.67)$$

$$\bar{\psi} = H(\cosh \alpha_3 Z + \frac{\mu_3^*}{\delta_1\theta_1} \sinh \alpha_3 Z - \cos \beta_3 Z + \mu_3^* \sin \beta_3 Z) \qquad (8.68)$$

where

$$\mu_3^* = \frac{-(\delta_1 \sinh \alpha_3 + \sin \beta_3)}{(1/\theta_1) \cosh \alpha_3 + \cos \beta_3} \tag{8.69}$$

The modal functions for the second set are:

$$\bar{\phi} = D(\cos \alpha_3' Z - \cot \alpha_3' \sin \alpha_3' Z - \cos \beta_3 Z + \cot \beta_3 \sin \beta_3 Z) \tag{8.70}$$

$$\bar{\psi} = H(\cos \alpha_3' Z - \frac{\eta_3^*}{\delta_2 \theta_2} \sin \alpha_3' Z - \cos \beta_3 Z + \eta_3^* \sin \beta_3 Z) \tag{8.71}$$

where

$$\eta_3^* = \frac{\delta_2 \sin \alpha_3' - \sin \beta_3}{(1/\theta_2) \cos \alpha_3' + \cos \beta_3} \tag{8.72}$$

8.6.4. CANTILEVER BEAM WITH ONE END FIXED AND FREE AT THE OTHER:

For a cantilever beam built in rigidly at the end $Z = 0$ so that warping is completely prevented, and with a free end at $Z = 1$, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \qquad \text{at} \quad Z = 0$$

and

$$\bar{\psi}' = 0, \ \left[s^2(K^2 - \Delta^2) + 1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \ \text{at} \ Z = 1.$$

The frequency equation for the first set, in this case, can be obtained as:

$$2 + \frac{(1+\theta_1^2)}{\theta_1} \cosh \alpha_3 \cos \beta_3 - \frac{(1-\delta_1^2)}{\delta_1} \sinh \alpha_3 \sin \beta_3 = 0 \tag{8.73}$$

The frequency equation for the second set is given by:

$$2 + \frac{(1+\theta_2^2)}{\theta_2} \cos \alpha_3' \cos \beta_3 - \frac{(1+\delta_2^2)}{\delta_2} \sin \alpha_3' \sin \beta_3 = 0 \qquad (8.74)$$

The modal functions for the first set are:

$$\bar{\phi} = D(\cosh \alpha_3 Z - \delta_1 \theta_1 \eta_4^* \sinh \alpha_3 Z - \cos \beta_3 Z + \eta_4^* \sin \beta_3 Z) \qquad (8.75)$$

$$\bar{\psi} = H(\cosh \alpha_3 Z + \frac{\mu_4^*}{\delta_1 \theta_1} \sinh \alpha_3 Z - \cos \beta_3 Z + \mu_4^* \sin \beta_3 Z) \qquad (8.76)$$

where

$$\eta_4^* = \frac{(1/\delta_1) \sinh \alpha_3 - \sin \beta_3}{\theta_1 \cosh \alpha_3 + \cos \beta_3} \qquad (8.77)$$

$$\mu_4^* = -\frac{(\delta_1 \sinh \alpha_3 + \sin \beta_3)}{(1/\theta_1) \cosh \alpha_3 + \cos \beta_3} \qquad (8.78)$$

The modal functions for the second set are:

$$\bar{\phi} = D(\cos \alpha_3' Z + \delta_2 \theta_2 \eta_5^* \sin \alpha_3' Z - \cos \beta_3 Z + \eta_5^* \sin \beta_3 Z) \qquad (8.79)$$

$$\bar{\psi} = H(\cos \alpha_3' Z - \frac{\mu_5^*}{\delta_2 \theta_2} \sin \alpha_3' Z - \cos \beta_3 Z + \mu_5^* \sin \beta_3 Z) \qquad (8.80)$$

where

$$\eta_5^* = \frac{(1/\delta_2) \sin \alpha_3' - \sin \beta_3}{\theta_2 \cos \alpha_3' + \cos \beta_3} \qquad (8.81)$$

$$\mu_5^* = \frac{\delta_2 \sin \alpha_3' - \sin \beta_3}{(1/\theta_2) \cos \alpha_3' + \cos \beta_3} \qquad (8.82)$$

8.6.5. CANTILEVER BEAM WITH ONE END SIMPLY SUPPORTED AND FREE AT THE OTHER:

For a cantilever beam simply supported at the end Z = 0 and free at Z = 1, the boundary conditions are:

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 0,$$

and

$$\bar{\psi}' = 0, \; \left[s^2(K^2 - \Delta^2) + 1 \right] \bar{\phi}' - (2L/h)\,\bar{\psi} = 0 \text{ at } Z = 1.$$

The frequency equation for the first set, in this case becomes:

$$\delta_1 \tanh \alpha_3' - \theta_1 \tan \beta_3 = 0 \tag{8.83}$$

The frequency equation for the second set is given by:

$$\delta_2 \tan \alpha_3' + \theta_2 \tan \beta_3 = 0 \tag{8.84}$$

The modal functions for the first set are:

$$\bar{\phi} = \frac{\delta_1 \cos \beta_3}{\cosh \alpha_3} \sinh \alpha_3 Z + \sin \beta_3 Z \tag{8.85}$$

$$\bar{\psi} = \frac{\sin \beta_3}{\delta_1 \sinh \alpha_3} \cosh \alpha_3 Z + \cos \beta_3 Z \tag{8.86}$$

The modal functions for the second set can be obtained as:

$$\bar{\phi} = - \frac{\delta_2 \cos \beta_3}{\cos \alpha_3'} \sin \alpha_3' Z + \sin \beta_3 Z \tag{8.87}$$

$$\bar{\psi} = - \frac{\sin \beta_3}{\delta_2 \sin \alpha_3'} \cos \alpha_3' Z + \cos \beta_3 Z \tag{8.88}$$

8.6.6. BEAM WITH FREE ENDS:

In the case of a beam which is free at both ends, the boundary conditions are:

$$\bar{\psi}' = 0, \quad \left[s^2(K^2-\Delta^2)+1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \text{ at } Z = 0,$$

and

$$\bar{\psi}' = 0, \quad \left[s^2(K^2-\Delta^2)+1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \text{ at } Z = 1.$$

The frequency equation for the first set, in this case ca be obtained as:

$$2-2 \cosh \alpha_3 \cos \beta_3 + \frac{(\theta_1^2-\delta_1^2)}{\delta_1 \theta_1} \sinh \alpha_3 \sin \beta_3 = 0 \qquad (8.89)$$

The frequency equation for the second set is given by:

$$2-2 \cos \alpha_3' \cos \beta_3 + \frac{(\theta_2^2+\delta_2^2)}{\delta_2^2 \theta_2} \sin \alpha_3' \sin \beta_3 = 0 \qquad (8.90)$$

The modal functions for the first set can be obtained as:

$$\bar{\phi} = D(\cosh \alpha_3 Z + \eta_6^* \delta_1 \sinh \alpha_3 Z + (1/\theta_1)\cos \beta_3 Z + \eta_6^* \sin \beta_3 Z) \qquad (8.91)$$

$$\bar{\psi} = H(\cosh \alpha_3 Z - \frac{\eta_6^*}{\delta_1} \sinh \alpha_3 Z + \theta_1 \cos \beta_3 Z + (1/\eta_6^*)\sin \beta_3 Z) \qquad (8.92)$$

where

$$\eta_6^* = \frac{\cosh \alpha_3 - \cos \beta_3}{\delta_1 \sinh \alpha_3 - \theta_1 \sin \beta_3} \qquad (8.93)$$

The modal functions for the second set are given by:

$$\bar{\phi} = D(\cos \alpha_3' Z - \delta_2 \mu_6^* \sin \alpha_3' Z + (1/\theta_2) \cos \beta_3 Z + \mu_6^* \sin \beta_3 Z) \qquad (8.94)$$

$$\bar{\psi} = H(\cos \alpha_3' Z - (\mu_6^*/\delta_2) \sin \alpha_3' Z + \theta_2 \cos \beta_3 Z + (1/\mu_6^*) \sin \beta_3 Z) \qquad (8.95)$$

where

$$\mu_6^* = - \frac{\cos \alpha_3' - \cos \beta_3}{\delta_2 \sin \alpha_3' + \theta_2 \sin \beta_3} \qquad (8.96)$$

8.7. APPROXIMATE SOLUTIONS BY GALERKIN'S TECHNIQUE:

Except for the simply supported beam, the frequency equations for other boundary conditions derived in the section (8.6) can be observed to be highly transcendental and are solved on a digital computer only by lengthy trial-and-error method. An attempt has been made in this section to derive approximate expressions for the torsional frequencies and buckling loads of fixed-fixed beam and of a beam fixed at one end and simply supported at the other, utilizing the Galerkin's technique.

8.7.1. FIXED-FIXED BEAM:

To satisfy the boundary conditions in this case, the normal function of angle of twist $\bar{\phi}$ can be assumed in the form

$$\bar{\phi} = \sum_{n=1}^{\infty} B_n (1 - \cos 2 n\pi Z) \qquad (8.97)$$

Substituting Eq.(8.97) in the differential Equation (8.20) and using the Galerkin's technique, expression for the frequency parameter λ^2, in this can be obtained as:

$$3\lambda^4 s^2 d^2 - \lambda^2 \left\{ 3 + 4n^2\pi^2 \left[s^2 + d^2 + s^2 d^2 (K^2 - \Delta^2) \right] + 12 s^2 d^2 \gamma^2 \right\}$$
$$+ \left\{ 16 n^4 \pi^4 \left[s^2 (K^2 - \Delta^2) + 1 \right] + 4n^2\pi^2 (K^2 - \Delta^2) + 4\gamma^2 (3 + 4n^2\pi^2 s^2) \right\} = \qquad (8.98)$$

Eq.(8.98) gives two real positive roots given by

$$\lambda^2_{mn} = \frac{1}{3s^2d^2} \left[3 + 4n^2\pi^2 \left[s^2 + d^2 + s^2 d^2 (K^2 - \Delta^2) \right] + 12 s^2 d^2 \gamma^2 \right]$$
$$+ (-1)^m \left[\left\{ 3 + 4n^2\pi^2 \left[s^2 + d^2 + s^2 d^2 (K^2 - \Delta^2) \right] + 12 s^2 d^2 \gamma^2 \right\}^2 \right.$$
$$- 12 s^2 d^2 \left\{ 16 n^4 \pi^4 | s^2 (K^2 - \Delta^2) + 1 | + 4n^2\pi^2 (K^2 - \Delta^2) \right.$$
$$\left. \left. + 4\gamma^2 (3 + 4n^2\pi^2 s^2) \right\} \right]^{1/2} \Bigg\} \qquad (8.99)$$

For a beam not vibrating, ie., $\lambda = 0$, the expression for the buckling load can be obtained from Eq.(8.98) as

$$\Delta^2_{cr} = K^2 + \left[\frac{4\pi^4 + \gamma^2 (3 + 4\pi^2 s^2)}{\pi^2 (1 + 4\pi^2 s^2)} \right] \qquad (8.10$$

If the effect of shear deformation is neglected, ie., $s^2 = 0$, Eq.(8.100) reduces to:

$$\Delta^2_{cr} = 4\pi^2 + K^2 + (3/\pi^2)\gamma^2 \qquad (8.10$$

which is same as Eq.(2.74) obtained by utilizing Timoshenko torsion theory.

If the effects of longitudinal inertia and shear deformation are neglected, ie., $s^2 = d^2 = 0$, Eq.(8.98) yields:

$$\lambda = 2\left[(n^2\pi^2/3)(4\,n^2\pi^2 + K^2 - \triangle^2) + \gamma^2\right]^{1/2} \tag{8.102}$$

which is same as Eq.(2.73).

8.7.2. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

To satisfy the boundary conditions in this case, the normal function of angle of twist $\bar{\phi}$ can be taken as:

$$\bar{\phi} = \sum_{n=1}^{\infty} D_n\left(\cos\frac{n\pi}{2}z - \cos\frac{3n\pi}{2}z\right) \tag{8.103}$$

Substituting Eq.(8.103) in the differential Equation (8.20) and using the Galerkin's technique, the expression for the Frequency parameter λ^2, in this case can be obtained as:

$$16\lambda^4 s^2 d^2 - \lambda^2\left\{16 + 20\,n^2\pi^2\left[s^2 + d^2 + s^2 d^2(K^2 - \triangle^2)\right] + 64\,s^2 d^2\gamma^2\right\}$$
$$+\left\{41\,n^4\pi^4\left[s^2(K^2 - \triangle^2) + 1\right] + 20\,n^2\pi^2(K^2 - \triangle^2) + 16\gamma^2(4 + 5\,n^2\pi^2 s^2)\right\} = \tag{8.104}$$

From Eq.(8.104) we have

$$\lambda^2_{mn} = \frac{1}{16\,s^2 d^2}\left\{\left[16 + 20n^2\pi^2\left[s^2 + d^2 + s^2 d^2(K^2 - \triangle^2)\right] + 64\,s^2 d^2\gamma^2\right.\right.$$
$$+ (-1)^m\left[\left\{16 + 20n^2\pi^2\left[s^2 + d^2 + s^2 d^2(K^2 - \triangle^2)\right] + 64\,s^2 d^2\gamma^2\right\}^2\right.$$
$$- 64\,s^2 d^2\left\{41n^4\pi^4\left[s^2(K^2 - \triangle^2) + 1\right] + 20\,n^2\pi^2(K^2 - \triangle^2)\right.$$
$$\left.\left.+ 16\gamma^2(4 + 5\,n^2\pi^2 s^2)\right\}\right]^{1/2} \tag{8.105}$$

For a beam not vibrating, ie., $\lambda = 0$, and the expression for the buckling load can be obtained from Eq.(8.104) as:

$$\Delta_{cr}^{2} = K^{2} + \left[\frac{2.05\ \pi^{4} + 0.8\gamma^{2}(4+5\ \pi^{2}s^{2})}{\pi^{2}(1+2.05\ \pi^{2}s^{2})} \right] \qquad (8.106)$$

If the effect of shear deformation is neglected, ie., $s^2 = 0$, Eq.(8.106) reduces to:

$$\Delta_{cr}^{2} = 2.05\ \pi^{2} + K^{2} + (3.2/\pi^{2})\gamma^{2} \qquad (8.107)$$

which is same as Eq.(2.77) derived by utilizing Timoshenko torsion theory.

If the effects of longitudinal inertia and shear deformation are neglected, ie., $s^2 = d^2 = 0$, Eq.(8.104) yields:

$$\lambda = \left[1.25\ n^{2}\pi^{2}(2.05\ n^{2}\pi^{2} + K^{2} - \Delta^{2}) + 4\gamma^{2} \right]^{1/2} \qquad (8.108)$$

which is same as Eq.(2.76).

8.8. LIMITING CONDITIONS:

The limiting conditions at which the combined influence of the axial compressive load and elastic foundation on the torsional frequency becomes zero, for some cases are as follows:

(1) Simply-Supported Beam:

From Eq.(8.44) we get two limiting conditions in this case. They are:

(a) $sd\gamma = 0.5\ n\pi\Delta$ (8.109)

(b) $\gamma = 0.5\ n\pi\Delta$ (8.110)

(2) Fixed-Fixed Beam: From Eq.(8.98) the limiting conditions in this case are:

$$(a)\ \sqrt{3}\, sd\,\nu = n\pi\Delta \tag{8.111}$$

$$(b)\ \nu = n\pi\Delta \left[\frac{1+4\, n^2\pi^2 s^2}{3+4\, n^2\pi^2 s^2}\right]^{1/2} \tag{8.112}$$

(3) Beam fixed at one end and Simply supported at the other:

From Eq.(8.104) the limiting conditions in this case are:

$$(a)\ 4\, sd\,\nu = \sqrt{5}\, n\pi\Delta \tag{8.113}$$

$$(b)\ \nu = 0.559\, n\pi\Delta \left[\frac{1+2.05\, n^2\pi^2 s^2}{1+1.25\, n^2\pi^2 s^2}\right]^{1/2} \tag{8.114}$$

If the effect of shear deformation is neglected, ie., $s^2 = 0$, Eqs.(8.112) and (8.114) reduces to Eqs.(2.79) and (2.80) derived previously.

For the above relations in various cases between ν and Δ there will be no influence of axial load and elastic foundation on the torsional frequency of vibration. This can be observed to be due to the opposite nature of their individual effects and these individual effects get nullified at these limiting conditions for various cases.

8.9. RESULTS AND CONCLUSIONS:

In this section, the results obtained on IBM 1130 Computer are presented in Tables 8.1 to 8.16 to show the effects of various non-dimensional parameters on the buckling loads and torsional frequencies of simply supported, clamped-clamped and clamped-simply supported beams resting on elastic foundation. Extensive design data ~~is~~ are made available in these tables. The main interest is to find the influences of shear deformation and longitudinal inertia on the frequencies of vibration of a short thin-walled beam resting on continuous elastic foundation and subjected to an axial compressive load.

The values of the torsional buckling load Δ_{cr} for the three boundary conditions are given in Table 8.1 for various values of the warping parameter K and shear parameter s. It is well known that the effect of increase in the value of K is to increase the buckling load considerably. From Table 8.1, we observe that for any constant value of K, the effect of increase in the value of s is to decrease the torsional buckling load, and that this reduction becomes significant for values of $K \leq 1$. Also, the effect of shear deformation in reducing the buckling load is comparitively considerable in clamped-clamped beams than in other cases.

The results showing the combined effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) are given in Tables 8.2, 8.6 and 8.10, for values of K = 0.01 and s = 2d. The percentage

T A B L E - 8.1

Effects of shear deformation and elastic foundation on the torsional buckling loads of simply supported, clamped-clamped and clamped-simply supported thin-walled beams of open section.

γ	s	Simply supported beam			Clamped-clamped beam			Clamped-simply supported beam		
		K=0.01	K=1.00	K=10.00	K=0.01	K=1.00	K=10.00	K=0.01	K=1.00	K=10.00
0	0.04	3.117	3.274	10.474	6.094	6.175	11.710	4.427	4.539	10.936
	0.08	3.047	3.207	10.454	5.614	5.702	11.468	4.232	4.349	10.859
	0.10	2.997	3.160	10.440	5.320	5.413	11.327	4.102	4.222	10.809
4	0.04	4.025	4.147	10.780	6.466	6.542	11.908	4.972	5.072	11.168
	0.08	3.971	4.098	10.760	5.977	6.060	11.650	4.782	4.886	11.085
	0.10	3.933	4.058	10.746	5.679	5.766	11.500	4.656	4.762	11.031
8	0.04	5.971	6.054	11.647	7.471	7.538	12.483	6.332	6.411	11.836
	0.08	5.935	6.018	11.628	6.954	7.025	12.180	6.143	6.224	11.736
	0.10	5.909	5.993	11.616	6.640	6.715	12.004	6.018	6.100	11.671
12	0.04	8.251	8.311	12.964	8.898	8.954	13.385	8.107	8.168	12.873
	0.08	8.225	8.285	12.948	8.331	8.391	13.015	7.907	7.970	12.748
	0.10	8.206	8.267	12.936	7.988	8.051	12.799	7.775	7.839	12.667

TABLE - 8.2

Effects of axial compressive load longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams (ν= 0, K=0.01 and s=2d).

Δ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.5	0.00	0.00	94.944	1.000	1548.694	1.000	7868.009	1.000	24897.484	1.000
	0.04	0.02	92.977	0.990	1436.073	0.963	6702.987	0.923	19091.711	0.876
	0.08	0.04	87.927	0.962	1186.118	0.875	4726.499	0.775	11592.957	0.682
	0.10	0.05	84.469	0.943	1053.103	0.825	3900.204	0.704	9041 .121	0.603
1.0	0.00	0.00	87.541	1.000	1519.085	1.000	7801.389	1.000	24779.051	1.000
	0.04	0.02	85.784	0.990	1406.831	0.962	6638.099	0.922	18977.887	0.875
	0.08	0.04	80.626	0.960	1157.668	0.873	4663.736	0.773	11482.309	0.681
	0.10	0.05	77.213	0.939	1024.978	0.821	3837.396	0.701	8930.559	0.600
1.5	0.00	0.00	75.204	1.000	1469.737	1.000	7690.355	1.000	24581.656	1.000
	0.04	0.02	73.329	0.987	1358.195	0.961	6530.017	0.921	18788.234	0.874
	0.08	0.04	68.469	0.954	1110.274	0.869	4559.131	0.770	11297.918	0.678
	0.10	0.05	65.126	0.931	978.090	0.816	3734.035	0.697	8746.240	0.596
2.0	0.00	0.00	57.932	1.000	1400.649	1.000	7534.908	1.000	24305.309	1.000
	0.04	0.02	56.267	0.986	1290.017	0.960	6378.824	0.920	18522.688	0.873
	0.08	0.04	51.430	0.942	1043.903	0.863	4412.663	0.765	11039.711	0.674
	0.10	0.05	48.202	0.912	912.452	0.807	3588.589	0.690	8468.141	0.591
2.5	0.00	0.00	35.726	1.000	1311.822	1.000	7335.048	1.000	23950.000	1.000
	0.04	0.02	33.949	0.775	1202.360	0.957	6184.101	0.918	18181.340	0.871
	0.08	0.04	29.521	0.909	958.557	0.855	4224.322	0.759	10707.641	0.669
	0.10	0.05	26.436	0.860	828.042	0.794	3401.540	0.681	8156.134	0.584
3.0	0.00	0.00	8.584	1.000	1203.256	1.000	7090.774	1.000	23515.734	1.000
	0.04	0.02	7.293	0.922	1095.507	0.954	5946.624	0.916	17764.387	0.869
	0.08	0.04	2.760	0.567	854.275	0.843	3994.094	0.751	10301.721	0.662

T A B L E - 8.3

Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams (Λ= 0, K=0.01, s=2d.

δ γ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
2	0.00	0.00	113.411	1.000	1574.563	1.000	7906.216	1.000	24952.965	1.000
	0.04	0.02	111.327	0.991	1461.376	0.963	6740.149	0.923	19144.809	0.876
	0.08	0.04	106.146	0.967	1210.974	0.877	4762.502	0.776	11644.762	0.683
	0.10	0.05	102.556	0.951	1077.675	0.827	3935.933	0.706	9092.303	0.604
4	0.00	0.00	161.411	1.000	1622.563	1.000	7954.216	1.000	25000.965	1.000
	0.04	0.02	159.197	0.993	1508.751	0.964	6786.883	0.924	19190.977	0.876
	0.08	0.04	153.472	0.975	1257.070	0.880	4807.706	0.777	11689.590	0.683
	0.10	0.05	149.588	0.963	1123.266	0.832	3980.823	0.707	9137.707	0.605
6	0.00	0.00	241.411	1.000	1702.563	1.000	8034.216	1.000	25080.965	1.000
	0.04	0.02	238.979	0.995	1587.796	0.966	6864.851	0.924	19267.863	0.876
	0.08	0.04	232.365	0.981	1333.911	0.885	4883.068	0.780	11764.303	0.685
	0.10	0.05	227.977	0.972	1199.249	0.839	4055.624	0.710	9212.363	0.606
8	0.00	0.00	353.411	1.000	1814.563	1.000	8146.216	1.000	25192.965	1.000
	0.04	0.02	350.677	0.996	1698.340	0.967	6973.980	0.925	19375.590	0.877
	0.08	0.04	342.828	0.985	1441.493	0.891	4988.573	0.782	11868.906	0.686
	0.10	0.05	337.704	0.978	1305.604	0.848	4160.333	0.715	9316.879	0.608
10	0.00	0.00	497.411	1.000	1958.563	1.000	8290.217	1.000	25336.965	1.000
	0.04	0.02	493.929	0.996	1840.202	0.969	7113.926	0.926	19513.777	0.878
	0.08	0.04	484.814	0.987	1579.773	0.898	5124.179	0.786	12003.357	0.688
	0.10	0.05	478.771	0.981	1442.318	0.858	4294.930	0.720	9451.234	0.611
12	0.00	0.00	673.411	1.000	2134.563	1.000	8466.217	1.000	25512.965	1.000
	0.04	0.02	669.471	0.997	2013.930	0.971	7285.458	0.928	19683.035	0.878
	0.08	0.04	658.378	0.988	1748.797	0.905	5289.933	0.791	12167.723	0.691
	0.10	0.05	651.147	0.983	1609.352	0.868	4459.380	0.726	9615.397	0.614

TABLE - 8.4

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, I Mode				Values of λ^2, II Mode			
			γ / 0	4	8	12	γ / 0	4	8	12
0.0	0.00	0.00	97.411	161.411	353.411	673.411	1558.563	1622.563	1814.563	2134.563
	0.04	0.02	95.559	159.197	350.677	669.471	1445.771	1508.751	1698.340	2013.930
	0.08	0.04	90.361	153.472	342.827	658.378	1195.602	1257.070	1441.493	1748.797
	0.10	0.05	86.882	149.588	337.704	651.147	1062.477	1123.266	1305.604	1609.352
1.0	0.00	0.00	87.541	151.541	343.541	663.541	1519.085	1583.085	1775.085	2095.085
	0.04	0.02	85.784	149.421	340.902	659.693	1406.831	1469.809	1659.398	1974.985
	0.08	0.04	80.626	143.735	333.092	648.645	1157.668	1219.141	1403.570	1710.881
	0.10	0.05	77.213	139.921	328.039	641.484	1024.978	1085.768	1268.116	1571.888
2.0	0.00	0.00	57.932	121.932	313.932	633.932	1400.649	1464.649	1656.649	1976.649
	0.04	0.02	56.267	119.904	311.383	630.172	1290.017	1352.993	1542.576	1858.151
	0.08	0.04	51.430	114.541	303.900	619.453	1043.303	1105.378	1289.820	1597.157
	0.10	0.05	48.202	110.912	299.033	612.487	912 .452	973.256	1155.642	1459.477
3.0	0.00	0.00	8.584	72.584	264.584	584.584	1203.256	1267.256	1459.256	1779.256
	0.04	0.02	7.293	70.481	262.407	581.189	1095.507	1158.481	1348.052	1663.609
	0.08	0.04	2.760	65.873	255.233	570.791	854.275	915.757	1100.220	1407.593
	0.10	0.05	0.000	62.552	250.682	564.149	-	785.684	968. 132	1272.070

T A B L E - 8.5

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, III Mode				Values of λ^2, IV Mode			
			γ 0	4	8	12	γ 0	4	8	12
0.0	0.00	0.00	7890.216	7954.216	8146.216	8466.217	24936.965	25000.965	25192.965	25512.965
	0.04	0.02	6724.678	6786.883	6973.380	7285.458	19129.629	19190.977	19375.590	19683.035
	0.08	0.04	4747.425	4807.706	4988.573	5289.933	11629.818	11689.590	11868.906	12167.723
	0.10	0.05	3920.978	3980.823	4160.333	4459.380	9077.973	9137.707	9316.879	9615.397
1.0	0.00	0.00	7801.389	7865.389	8057.389	8377.389	24779.051	24843.051	25035.051	25355.051
	0.04	0.02	6638.099	6700.221	6887.314	7198.785	18977.887	19039.234	19223.910	19531.270
	0.08	0.04	4663.736	4724.018	4904.900	5206.283	11482.309	11542.084	11721.424	12020.270
	0.10	0.05	3837.896	3897.752	4077.286	4376.382	8390.559	8990.305	9169.510	9468.094
2.0	0.00	0.00	7534.908	7598.908	7790.908	8110.908	24305.309	24369.309	24561.309	24881.309
	0.04	0.02	6378.824	6441.023	6628.103	6939.555	18522.688	18584.102	18768.758	19076.086
	0.08	0.04	4412.663	4472.961.	4653.874	4955.313	11039.711	11099.510	11278.904	11577.842
	0.10	0.05	3588.589	3648.473	3828.089	4127.317	8488.141	8547.924	8727.244	9026.012
3.0	0.00	0.00	7090.774	7154.774	7346.774	7666.774	23515.734	23579.734	23771.734	24091.734
	0.04	0.02	5946.624	6008.814	6195.874	6507.368	17764.387	17825.793	18010.414	18317.836
	0.08	0.04	3994.094	4054.407	4235.383	4536.924	10501.721	10361.549	10541.033	10840.133
	0.10	0.05	-	3232.762	3412.513	3711.965	-	7810.000	7989.504	8263.578

T A B L E - 8.6

Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported thin-walled beams ($\gamma = 0$, $K=0.01$, $s=2d$).

Δ	s	d	Values of λ^2 and $q = \lambda/\lambda_0$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.0	0.00	0.00	249.614	1.000	3993.813	1.000	20218.664	1.000	63900.938	1.000
	0.04	0.02	243.820	0.988	3642.962	0.955	16690.797	0.909	46820.211	0.856
	0.08	0.04	227.685	0.955	2962.263	0.856	11414.037	0.751	27857.102	0.660
	0.10	0.05	217.290	0.933	2572.443	0.803	9390.227	0.681	21881.023	0.585
2.0	0.00	0.00	200.266	1.000	3796.419	1.000	19774.527	1.000	63111.367	1.000
	0.04	0.02	194.088	0.984	3439.561	0.952	16216.939	0.906	45940.109	0.853
	0.08	0.04	176.847	0.940	2706.261	0.844	10864.486	0.741	26763.426	0.651
	0.10	0.05	165.634	0.909	2341.124	0.785	8792.682	0.667	20658.918	0.572
4.0	0.00	0.00	52.221	1.000	3204.241	1.000	18442.125	1.000	60742.649	1.000
	0.04	0.02	44.890	0.927	2829.545	0.940	14796.113	0.896	43300.180	0.844
	0.08	0.04	24.331	0.683	2046.722	0.799	9220.086	0.707	23502.168	0.622
	0.10	0.05	10.695	0.453	1648.723	0.717	7013.864	0.617	17055.133	0.530

T A B L E - 8.7

Effects of elastic foundation, longitudinal inertia, and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported short thin-walled beams ($\Lambda = 0$, K=0.01, s=2d).

γ	s	d	Values of λ^2 and $q = \lambda/\lambda_0$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0	0.00	0.00	249.614	1.000	3993.813	1.000	20218.664	1.000	63900.938	1.000
	0.04	0.02	243.820	0.988	3642.962	0.955	16690.797	0.909	46820.211	0.856
	0.08	0.04	227.685	0.955	2926.263	0.856	11414.037	0.751	27857.102	0.660
	0.10	0.05	217.290	0.933	2572.443	0.803	9390.227	0.681	21881.023	0.585
4	0.00	0.00	313.614	1.000	4057.813	1.000	20282.664	1.000	63964.938	1.000
	0.04	0.02	307.523	0.990	3705.920	0.956	16753.008	0.909	46881.867	0.856
	0.08	0.04	290.666	0.963	2987.917	0.858	11475.691	0.752	27920.129	0.661
	0.10	0.05	279.870	0.945	2633.897	0.806	9452.793	0.683	21946.465	0.586
8	0.00	0.00	505.614	1.000	4249.813	1.000	20474.664	1.000	64156.938	1.000
	0.04	0.02	498.630	0.993	3894.979	0.957	16939.648	0.910	47066.828	0.857
	0.08	0.04	479.608	0.974	3172.854	0.864	11660.617	0.755	28109.188	0.662
	0.10	0.05	467.568	0.962	2818.238	0.814	9640.490	0.686	22142.805	0.587
12	0.00	0.00	825.614	1.000	4569.813	1.000	20794.664	1.000	64476.938	1.000
	0.04	0.02	817.143	0.995	4209.766	0.960	17250.523	0.911	47375.094	0.857
	0.08	0.04	794.477	0.981	3481.041	0.873	11968.803	0.759	28424.301	0.664
	0.10	0.05	780.330	0.972	3125.369	0.827	9953.275	0.692	22470.098	0.590

T A B L E - 8.8

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of clamped-simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, I Mode				Values of λ^2, II Mode			
			γ 0	4	8	12	γ 0	4	8	12
0.0	0.00	0.00	249.614	313.614	505.614	825.614	3993.813	4057.813	4249.813	4569.813
	0.04	0.02	243.820	307.523	498.630	817.143	3642.962	3705.920	3894.979	4209.766
	0.08	0.04	227.685	290.666	479.608	794.477	2926.263	2987.917	3172.854	3481.041
	0.10	0.05	217.290	279.870	467.568	780.330	2572.443	2633.897	2818.238	3125.369
2.0	0.00	0.00	200.266	264.266	456.266	776.266	3796.419	3860.419	4052.419	4372.420
	0.04	0.02	194.088	257.790	448.898	767.410	3439.562	3502.519	3691.578	4006.365
	0.08	0.04	176.847	239.827	428.758	743.626	2706.261	2767.915	2952.817	3260.980
	0.10	0.05	165.634	228.210	415.907	728.660	2341.124	2042.555	2586.824	2893.826
4.0	0.00	0.00	52.221	116.221	308.221	628.221	3204.241	3268.241	3460.241	3780.241
	0.04	0.02	44.890	108.592	299.700	618.212	2829.545	2892.503	3081.375	3396.348
	0.08	0.04	24.331	87.300	276.242	591.099	2046.722	2108.340	2293.136	2601.243
	0.10	0.05	10.695	73.266	260.949	573.678	1648.723	1610.082	1894.137	2200.772

TABLE - 8.9

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of clamped-simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, III Mode				Values of λ^2, IV Mode			
		γ	0	4	8	12	0	4	8	12
0.0	0.00	0.00	20218.664	20282.664	20474.664	20794.664	63900.938	63964.938	64156.938	64476.938
	0.04	0.02	16690.767	16753.008	16939.648	17250.523	46820.211	46881.867	47066.828	47375.034
	0.08	0.04	11414.037	11475.691	11660.617	11968.803	27857.102	27920.129	28109.188	28424.301
	0.10	0.05	9390.227	8452.793	9640.490	9953.275	21881.023	21946.465	22142.805	22470.098
2.0	0.00	0.00	19774.527	19838.527	20030.527	20350.527	63111.367	63175.367	63367.367	63687.367
	0.04	0.02	16216.939	16279.152	16465.789	16776.852	45940.109	46001.766	46186.727	46494.805
	0.08	0.04	10864.486	10926.117	11110.938	11418.973	26763.426	26826.336	27015.141	27329.809
	0.10	0.05	8792.682	8855.143	9042.502	9354.711	20658.918	20724.051	20919.469	21245.150
4.0	0.00	0.00	18442.125	18506.125	18698.125	19018.125	60742.649	60806.649	60998.649	61318.649
	0.04	0.02	14796.113	14858.139	15044.775	15355.838	43300.180	43361.836	43546.797	43855.063
	0.08	0.04	9220.086	9281.611	9466.164	9773.723	23502.168	23564.844	23752.856	24066.199
	0.10	0.05	7013.864	7076.006	7262.415	7573.032	17055.133	17119.438	17312.328	17633.840

T A B L E - 8.10

Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled beams ($\gamma = 0$, $K=0.01$, $s=2d$).

Δ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.0	0.00	0.00	519.521	1.000	8312.322	1.000	42081.117	1.000	132997.094	1.000
	0.04	0.02	506.516	0.987	7553.774	0.953	34643.352	0.907	97904.031	0.858
	0.08	0.04	472.111	0.953	6119.002	0.858	24856.652	0.769	66324.172	0.706
	0.10	0.05	450.494	0.931	5463.667	0.811	21719.863	0.718	66035.985	0.705
2.0	0.00	0.00	466.883	1.000	8101.770	1.000	41607.375	1.000	132154.875	1.000
	0.04	0.02	452.002	0.984	7313.990	0.950	34029.055	0.904	96638.172	0.855
	0.08	0.04	412.165	0.940	5802.740	0.846	23865.852	0.757	63592.719	0.694
	0.10	0.05	386.737	0.910	5093.349	0.793	20378.367	0.700	60305.024	0.676
4.0	0.00	0.00	308.969	1.000	7470.111	1.000	40186.141	1.000	129628.250	1.000
	0.04	0.02	288.338	0.966	6594.636	0.940	32187.020	0.895	92843.719	0.846
	0.08	0.04	232.373	0.867	4857.074	0.806	20935.410	0.722	55818.211	0.656
	0.10	0.05	195.653	0.796	3994.760	0.731	16570.820	0.642	44487.195	0.386

TABLE - 8.11

Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled beams ($\Delta = 0$, K=0.01, s=2d).

γ	s	d	Values of λ^2 and $q = \lambda / \lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0	0.00	0.00	519.521	1.000	8312.322	1.000	42081.117	1.000	132997.094	1.000
	0.04	0.02	506.516	0.987	7553.774	0.953	34643.352	0.907	97904.031	0.858
	0.08	0.04	472.111	0.953	6119.002	0.858	24856,652	0.769	66324.172	0.706
	0.10	0.05	450.494	0.931	5463.663	0.811	21719.863	0.718	66035.985	0.705
4	0.00	0.00	583.521	1.000	8376.322	1.000	42145.117	1.000	133061.094	1.000
	0.04	0.02	570.218	0.989	7616.856	0.954	34706.063	0.907	97966.360	0.858
	0.08	0.04	535.162	0.958	6181.943	0.859	24923.539	0.769	66404.719	0.706
	0.10	0.05	513.281	0.938	5527.894	0.812	21795.211	0.719	65822.531	0.703
8	0.00	0.00	775.521	1.000	8568.322	1.000	42337.117	1.000	133253.094	1.000
	0.04	0.02	761.326	0.991	7805.977	0.954	34893.945	0.908	98155.736	0.858
	0.08	0.04	724.305	0.966	6370.738	0.862	25124.254	0.770	66646.406	0.707
	0.10	0.05	701.615	0.951	5720.594	0.817	22021.434	0.721	65231.055	0.700
12	0.00	0.00	1095.521	1.000	8888.322	1.000	42657.117	1.000	133573.094	1.000
	0.04	0.02	1079.714	0.993	8121.261	0.956	35207.117	0.908	98470.391	0.859
	0.08	0.04	1039.543	0.974	6685.378	0.867	25458.801	0.773	67049.672	0.708
	0.10	0.05	1015.444	0.963	6041.766	0.824	22399.109	0.725	64369.406	0.694

T A B L E - 8.12

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of clamped-clamped short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, I Mode				Values of λ^2, II Mode			
			γ / 0	4	8	12	γ / 0	4	8	12
0.0	0.00	0.00	519.521	583.521	775.521	1095.521	8312.322	8376.322	8568.322	8888.322
	0.04	0.02	506.516	570.218	761.326	1079.714	7553.774	7616.856	7805.977	8121.261
	0.08	0.04	472.111	535.162	724.305	1039.543	6119.002	6181.943	6370.738	6685.378
	0.10	0.05	450.494	513.281	701.615	1015.444	5463.663	5527.894	5720.594	6041.766
2.0	0.00	0.00	466.883	530.883	722.883	1042.883	8101.770	8165.770	8357.770	8677.770
	0.04	0.02	452.002	515.580	706.688	1025.076	7313.990	7376.947	7566.192	7881.476
	0.08	0.04	412.165	475.208	664.344	979.558	5802.740	5865.620	6054.274	6368.666
	0.10	0.05	386.737	449.511	637.808	951.564	5093.349	5157.397	5349.538	5669.769
4.0	0.00	0.00	308.969	372.969	564.969	884.969	7470.111	7534.111	7726.111	8046.111
	0.04	0.02	288.338	351.916	543.148	861.536	6594.636	6657.594	6846.839	7161.998
	0.08	0.04	232.373	295.400	484.505	799.657	4857.074	4919.814	5108.020	5421.674
	0.10	0.05	195.653	258.385	446.558	760.114	3994.760	4058.287	4248.851	4566.443

T A B L E - 8.13

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of clamped-clamped short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, III Mode				Values of λ^2, IV Mode			
			γ 0	4	8	12	γ 0	4	8	12
0.0	0.00	0.00	42081.117	42145.117	42337.117	42657.117	132997.094	133061.094	133253.094	133573.09
	0.04	0.02	34643.352	34706.063	34893.945	35207.117	97904 .031	97966.860	98155.735	98470.39
	0.08	0.04	24856.652	24923.539	25124.254	25458.801	66324.172	66404 .719	66646.406	67049.67
	0.10	0.05	21719.863	21795.211	22021.434	22399.109	66035.065	65822.531	65231.055	64369.40
2.0	0.00	0.00	41607.375	41671.375	41863.375	42183.375	132154.875	132218.875	132410.875	132730.87
	0.04	0.02	34029.055	34091.766	34279.641	34592.938	96638.172	96701.125	96889.750	97204.54
	0.08	0.04	23865.852	23932.473	24132.395	24465.621	63592.719	63671.820	63909.094	64304.89
	0.10	0.05	20378.367	20452.328	20674.352	21044.898	60305.024	60477.586	61004.914	61920.90
4.0	0.00	0.00	40186.141	40250.141	40442.141	40762.141	129628.250	129692.250	129884.250	125993.18
	0.04	0.02	32187.020	32249.606	32437.484	32750.656	92843.719	92906.672	93095.297	87095.59
	0.08	0.04	20935.410	21001.320	21198.992	21528.492	55818.211	55893.649	56120.031	56497.59
	0.10	0.05	16570.820	16641.430	16853.352	17206.852	44487.195	44583.672	44873.789	45359.73

T A B L E - 8.14

Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported short thin-walled beams (s=0.10 and d=0.05).

Δ	γ	Values of $q = \lambda/\lambda_o$ for K=1.0				Values of $q = \lambda/\lambda_o$ for K=10.0			
		I Mode	II Mode	III Mode	IV Mode	I Mode	II Mode	III Mode	IV Mode
0.0	0	0.9487	0.8297	0.7084	0.6063	0.9973	0.9847	0.9572	0.9176
	4	0.9643	0.8357	0.7108	0.6075	0.9973	0.9848	0.9573	0.9177
	8	0.9779	0.8511	0.7178	0.6110	0.9976	0.9851	0.9577	0.9180
	12	0.9834	0.8703	0.7287	0.6167	0.9976	0.9855	0.9582	0.9185
1.5	0	0.9377	0.8203	0.7005	0.5996	0.9974	0.9845	0.9569	0.9170
	4	0.9604	0.8272	0.7031	0.6008	0.9974	0.9846	0.9570	0.9171
	8	0.9771	0.8444	0.7105	0.6045	0.9976	0.9849	0.9573	0.9175
	12	0.9832	0.8656	0.7220	0.6104	0.9977	0.9854	0.9579	0.9180
3.0	0	0.7180	0.7832	0.6734	0.4776	0.9974	0.9841	0.9557	0.9152
	4	0.9359	0.7937	0.6766	0.5790	0.9974	0.9842	0.9559	0.9154
	8	0.9740	0.8191	0.6857	0.5831	0.9976	0.9845	0.9562	0.9157
	12	0.9825	0.8496	0.6995	0.5898	0.9977	0.9850	0.9568	0.9162

TABLE - 8.15

Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported short thin-walled beams ($a=0.10$ and $d=0.05$).

Δ	γ	Values of $q = \lambda / \lambda_o$ for K=1.0				Values of $q = \lambda / \lambda_o$ for K=10.0			
		I Mode	II Mode	III Mode	IV Mode	I Mode	II Mode	III Mode	IV Mode
0	0	0.9330	0.8026	0.6815	0.5852	1.0091	0.9729	0.9001	0.8461
	4	0.9447	0.8057	0.6827	0.5857	1.0083	0.9730	0.9004	0.8465
	8	0.9616	0.8143	0.6862	0.5875	1.0062	0.9733	0.9011	0.8476
	12	0.9722	0.8270	0.6918	0.5903	1.0035	0.9738	0.9025	0.8495
2	0	0.9094	0.7853	0.6668	0.5721	1.0086	0.9699	0.8941	0.8347
	4	0.9293	0.7889	0.6681	0.5727	1.0077	0.9700	0.8944	0.8350
	8	0.9547	0.7990	0.6719	0.5746	1.0056	0.9703	0.8952	0.8361
	12	0.9689	0.8135	0.6780	0.5776	1.0030	0.9709	0.8967	0.8378
4	0	0.4526	0.7173	0.6167	0.5299	1.0067	0.9598	0.8754	0.8032
	4	0.7940	0.7234	0.6184	0.5306	1.0059	0.9600	0.8757	0.8035
	8	0.9201	0.7399	0.6232	0.5327	1.0038	0.9606	0.8767	0.8045
	12	0.9556	0.7630	0.6310	0.5363	1.0013	0.9615	0.8784	0.8061

TABLE - 8.16

Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled Beams (a=0.10 and d=0.05).

		Values of $q = \lambda/\lambda_o$ for K=1.0				Values of $q = \lambda/\lambda_o$ for K=10.0			
Δ	γ	I Mode	II Mode	III Mode	IV Mode	I Mode	II Mode	III Mode	IV Mode
0	0	0.9358	0.8150	0.7230	0.6891	1.0094	0.9991	0.9632	0.9177
	4	0.9418	0.8166	0.7237	0.6882	1.0090	0.9991	0.9632	0.9177
	8	0.9539	0.8212	0.7258	0.6855	1.0080	0.9990	0.9633	0.9178
	12	0.9645	0.8284	0.7292	0.6813	1.0066	0.9988	0.9635	0.9180
2	0	0.9159	0.7975	0.7045	0.6874	1.0091	0.9980	0.9615	0.9156
	4	0.9250	0.7992	0.7062	0.6884	1.0088	0.9979	0.9615	0.9157
	8	0.9424	0.8044	0.7074	0.6913	1.0077	0.9979	0.9616	0.9158
	12	0.9572	0.8124	0.7109	0.6966	1.0064	0.9973	0.9618	0.9159
4	0	0.8104	0.7370	0.6471	0.5919	1.0083	0.9944	0.9561	0.9094
	4	0.8428	0.7395	0.6479	0.5924	1.0079	0.9944	0.9562	0.9094
	8	0.8944	0.7469	0.6505	0.5939	1.0068	0.9944	0.9563	0.9095
	12	0.9296	0.7584	0.6546	0.5964	1.0055	0.9943	0.9565	0.9097

reductions in the torsional frequencies due to increase in the axial compressive load can be observed from these tables to be slightly higher than those when the effects are neglected.

The combined effect of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) are shown in Tables 8.3, 8.7 and 8.11 for values of K = 0.01 and s = 2d. From these results it can be noted that the percentage increase in the torsional frequencies due to elastic foundation is slightly more than those when the second order effects are neglected. The results presented in Tables 8.4, 8.5, 8.8, 8.9, 8.12 and 8.13 show the combined effects of axial compressive load and elastic foundation in combination with the effects of longitudinal inertia and shear deformation on the first and second, third and fourth torsional frequencies (first set) of simply supported, clamped-clamped and clamped-simply supported beams respectively. It can be observed from these tables that the combined effects are almost the algebroic sum of the individual influences of various effects on the torsional frequencies of vibration. The results for the modifying quotients for the first four modes of vibration for simply-supported, clamped-clamped, and clamped-simply supported beams are respectively presented in Tables 8.14, 8.15 and 8.16 for values of s = 0.10, d = 0.05 and for various values of $\triangle$, γ and K. From these results we observe that for any set of values of K and γ, the influence of increase in the values of $\triangle$ in the range 0.0 to 3.0 is to decrease the modifying quotients (i.e., to increase the second order effects on the frequencies of vibration) for various modes by about 25 percent. For any constant set of values of Δ and K, the effect of increase in the values of γ in the range 0 to 12 is to increase the modifying quotients (i.e., to decrease the second order effects on the frequencies of vibration) for various modes at the most by 15 percent. For constant values of Δ and γ, the effect of increasing the value of K from 1.0 to 10.0 is to increase the modifying quotients for various modes by about 10 percent.

It is also observed that, for constant values of K and γ, the reduction in the frequency of vibration at the first mode is quite considerable for values of Δ nearing Δ_{cr}. From the various results presented in this section, we can conclude that the effects of shear deformation and longitudinal inertia on the torsional frequencies at higher modes become increasingly important for a beam with smaller values of warping parameter K and foundation parameter γ and for larger values of $\Delta \leq \Delta_{cr}$.

CHAPTER - IX

FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS AND STABILITY OF SHORT THIN-WALLED BEAMS RESTING ON CONTINUOUS ELASTIC FOUNDATION*.

9.1. INTRODUCTION:

The problem of torsional vibrations and stability of lengthy thin-walled beams of open section resting on Winkler-type elastic foundation is solved in Chapter III utilizing finite-element method. The stiffness, stability and mass matrices derived therein, does not include the second order effects such as longitudinal inertia and shear deformation. These second order effects cannot be neglected in the case of short and deep thin-walled beams and, as is shown in Chapter IV, they drastically change the torsional frequencies at higher modes of vibration.

The present chapter, therefore, aims at extending the finite element method presented in Chapter III to include the effects of longitudinal inertia and shear deformation. New stiffness, stability coefficient and mass matrices for a short or deep thin-walled beam are developed in this Chapter, which include the effects of longitudinal inertia and shear deformation in addition to the effects of axial time-invariant compressive load and elastic foundation. The method developed herein is useful in analyzing both uniform and non-uniform beams with any complex boundary conditions. The new stiffness and stability coefficient matrices are made use of in conjunction with the consistant mass matrix for finding the torsional frequencies, buckling loads and mode shapes of short uniform thin-walled beams with various end conditions. Results obtained for the case of a simply supported beam by the finite element method are compared with the exact ones obtained in Chapter VIII and an excellent agreement is observed even for a coarse sub-division of the beam.

---

* A paper by the author based on the results from this Chapter is communicated to Journal of Applied Mechanics, Transactions of ASME, for publication. See Ref.(56)

9.2. MODIFIED STRAIN ENERGY EXPRESSION INCLUDING THE EFFECTS OF AXIAL LOAD AND ELASTIC FOUNDATION:

Substituting Eq.(5.1) into Eq.(8.1), the strain energy U_4, due to the Winkler-type elastic foundation can be written in a modified form as:

$$U_4 = \frac{1}{2} \int_0^L K_t(\phi_t + \phi_s)^2 \, dz \tag{9.1}$$

Utilizing Eqs.(5.14) and (9.1), the total strain energy U at any instant t including the effect of Winkler-type elastic foundation can be written in a modified form as:

$$\begin{aligned} U &= U_1 + U_2 + U_3 + U_4 \\ &= \frac{1}{2} \int_0^L \Big[GC_s\Big(\frac{\partial\phi_t}{\partial z} + \frac{\partial\phi_s}{\partial z}\Big)^2 + EC_w\Big(\frac{\partial^2\phi_t}{\partial z^2}\Big)^2 \\ &\qquad + K' A_f G \frac{h}{2}^2 \Big(\frac{\partial\phi_s}{\partial z}\Big)^2 + K_t(\phi_t + \phi_s)^2 \Big] \, dz \end{aligned} \tag{9.2}$$

Substituting Eq.(5.1) into Eq.(8.3) the potential energy, W, due to the time-invariant axial compressive load P can be written in a modified form as:

$$W = \frac{1}{2} \int_0^L \frac{PI_p}{A} \left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right)^2 dz \qquad (9.3)$$

The total kinetic energy, T_k, at any time t in the modified form is given by:

$$T_k = \frac{1}{2} \int_0^L \left[\rho I_p \left(\frac{\partial \phi_t}{\partial t} + \frac{\partial \phi_s}{\partial t}\right)^2 + \rho C_w \left(\frac{\partial^2 \phi_t}{\partial z \partial t}\right)^2 \right] dz \qquad (9.4)$$

which is same as Eq.(5.15).

9.3. MODIFIED NATURAL BOUNDARY CONDITIONS:

Except for the case of a free end, the boundary conditions for simply supported and fixed ends remain the same as those given by Eqs.(5.16) and (5.17).

For the case of a ''free end'', the modified natural boundary conditions for the present problem become:

$$\frac{\partial^2 \phi_t}{\partial z^2} = 0; \quad \left(GC_s - \frac{PI_p}{A}\right) \frac{\partial \phi_t}{\partial z} + \left(GC_s - \frac{PI_p}{A} + K' A_f G \, {\frac{h}{2}}^2\right) \frac{\partial \phi_s}{\partial z} = 0 \qquad (9.5)$$

9.4. DERIVATION OF ELEMENT MATRICES INCLUDING AXIAL LOAD, ELASTIC FOUNDATION AND SECOND ORDER EFFECTS:

The expressions for the strain energy U, potential energy W and, Kinetic energy T_k, given by Eqs.(9.2), (9.3) and (9.4) respectively, for an element of length, l, can be written as follows:

$$U = \frac{1}{2} \int_0^l \Big[GC_s(\phi_t' + \phi_s')^2 + EC_w(\phi_t'')^2$$

$$+ K'A_fG\,\frac{h^2}{2}(\phi_s')^2 + K_t(\phi_t + \phi_s)^2 \Big]\,dz \qquad (9.6)$$

$$W = \frac{1}{2} \int_0^l \frac{PI_p}{A} (\phi_t' + \phi_s')^2\,dz \qquad (9.7)$$

and

$$T_k = \frac{1}{2} \int_0^l \Big[\rho I_p(\dot{\phi}_t + \dot{\phi}_s)^2 + \rho C_w(\dot{\phi}_t')^2 \Big]\,dz \qquad (9.8)$$

Direct substitution of Eqs.(5.24) to (5.36) into Eqs.(9.6), (9.7) and (9.8) and the resulting expressions into Hamilton's principle, Eq.(3.34), yields (for the Nth element):

$$\bar{\delta} I_N = \bar{\delta} \int_{t_1}^{t_2} \Big\{ \rho I_p \Big[\int_0^l \dot{\bar{R}}_{tN}^T\, \bar{A}^T \bar{A}\, \dot{\bar{R}}_{tN}\,dz + \int_0^l \dot{\bar{R}}_{sN}^T\, \bar{A}^T \bar{A}\, \dot{\bar{R}}_{sN}\,dz$$

$$+ \int_0^l \dot{\bar{R}}_{tN}^T \bar{A}^T\, \bar{A}\, \dot{\bar{R}}_{sN}\,dz + \int_0^l \dot{\bar{R}}_{sN}^T\, \bar{A}^T\, \bar{A}\, \dot{\bar{R}}_{tN}\,dz \Big]$$

$$+ \frac{\rho C_w}{2} \int_0^l \dot{\bar{R}}_{tN}^T\, \bar{A}_1^T\, \bar{A}_1\, \dot{\bar{R}}_{tN}\,dz$$

$$- \frac{1}{2} \int_0^l \bar{R}_{tN}^T \Big[EC_w\, \bar{A}_2^T \bar{A}_2 + GC_s \bar{A}_1^T \bar{A}_1 + K_t \bar{A}^T \bar{A} \Big]\, \bar{R}_{tN}\,dz$$

$$- \frac{1}{2} \int_0^l \bar{R}_{sN}^T \Big[(GC_s + K'A_fG\,h^2/2)\, \bar{A}_1^T \bar{A}_1 + K_t \bar{A}^T \bar{A} \Big]\, \bar{R}_{sN}\,dz$$

$$- \frac{GC_s}{2} \Big[\int_0^l \bar{R}_{tN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{sN}\,dz + \int_0^l \bar{R}_{sN}^T\, \bar{A}_1^T\, \bar{A}_1\, \bar{R}_{tN}\,dz \Big]$$

$$-\frac{K_t}{2}\left[\int_0^1 \bar{R}_{tN}^T\,\bar{A}^T\bar{A}\,\bar{R}_{sN}\,dz + \int_0^1 \bar{R}_{sN}^T\,\bar{A}^T\bar{A}\,\bar{R}_{tN}\,dz\right]$$

$$+\frac{PI_p}{2A}\left[\int_0^1 \bar{R}_{tN}^T\,\bar{A}_1^T\,\bar{A}_1\,\bar{R}_{tN}\,dz + \int_0^1 \bar{R}_{sN}^T\,\bar{A}_1^T\,\bar{A}_1\,\bar{R}_{sN}\,dz\right.$$

$$\left. + \int_0^1 \bar{R}_{tN}^T\,\bar{A}_1^T\,\bar{A}_1\,\bar{R}_{sN}\,dz + \int_0^1 \bar{R}_{sN}^T\,\bar{A}_1^T\,\bar{A}_1\,\bar{R}_{tN}\,dz\right]\Bigg\}\,dt$$

$$= 0 \tag{9.9}$$

Eq.(9.9) can be written more concisely as follows:

$$\bar{\delta} I_N = \bar{\delta}\int_{t_1}^{t_2}\frac{1}{2}\Big[(\rho I_p L)\,\dot{\bar{q}}_N^T\,\bar{m}_N\,\dot{\bar{q}}_N - (EC_w/L^3)\,\bar{q}_N^T\,\bar{k}_N\,\bar{q}_N$$

$$+ (PI_p/AL)\,\bar{q}_N^T\,\bar{s}_N\,\bar{q}_N\Big]\,dt = 0 \tag{9.10}$$

In Eq.(9.10) the terms $(\rho I_p L)\,\bar{m}_N$, $(EC_w/L^3)\bar{k}_N$ and $(PI_p/AL)\bar{s}_N$ denote respectively the mass matrix $\bar{M}_N$, the stiffness matrix $\bar{K}_N$ and stability coefficient matrix $\bar{s}_N$ of the Nth element. The matrices $\bar{m}_N$ and $\bar{q}_N$ obtained herein are the same as Eqs.(5.41) and (5.43) respectively. The matrices $\bar{k}_N$ and $\bar{s}_N$ are as follows:

$$\bar{k}_N = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{21}^T \\ \bar{k}_{21} & \bar{k}_{22} \end{bmatrix} \tag{9.11}$$

where

$$\bar{k}_{11} = \begin{bmatrix} 12N^2 & & \text{Sym.} & \\ 6N & 4 & & \\ -12N^2 & -6N & 12N^2 & \\ 6N & 2 & -6N & 4 \end{bmatrix}$$

$$+\frac{K^2}{30N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}$$

$$+\frac{4\gamma^2}{420N^4}\begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (9.12)$$

$$\bar{K}_{21} = \frac{K^2}{30N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}$$

$$+\frac{4\gamma^1}{420N^4}\begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (9.13)$$

$$K_{22} = \frac{(s^2K^2+1)}{30\ s^2N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}$$

$$+\frac{4\gamma^2}{420N^4}\begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (9.14)$$

and

$$\bar{s}_N = \begin{bmatrix} \bar{s}_{11} & \bar{s}_{21}^T \\ \bar{s}_{21} & \bar{s}_{22} \end{bmatrix} \tag{9.15}$$

where

$$\bar{s}_{11} = \bar{s}_{21} = \bar{s}_{22} = \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \tag{9.16}$$

Following the procedure outlined in Chapters III and V, the equations of motion for the discretized system can now be obtained from Eq.(9.10) as follows:

$$[\bar{k}_N - \Delta^2 \bar{s}_N]\,[\bar{Q}_N] = \lambda^2\,[\bar{m}_N]\,[\bar{Q}_N] \tag{9.17}$$

where the non-dimensional parameters Δ^2 and λ^2 are given by Eqs.(3.47) and (3.48).

In a similar way the equations of equilibrium for the totally assembled beam can be obtained as:

$$[\bar{k} - \Delta^2 \bar{s}]\,[\bar{Q}] = \lambda^2\,[\bar{m}]\,[\bar{Q}] \tag{9.18}$$

where $\bar{k}$, $\bar{s}$, $\bar{m}$ and $\bar{Q}$ denote the totally assembled matrices corresponding to the element matrices $\bar{k}_N$, $\bar{s}_N$, $\bar{m}_N$ and $\bar{Q}_N$ defined previously.

9.5. RESULTS AND CONCLUSIONS:

Results for the first and second sets of values of λ^2 for various values of the axial load parameter Δ and foundation parameter γ for simply supported beams for values of K = 1.541, s = 0.046 and d = 0.023, are obtained on IBM 1130 Computer at Andhra University, Waltair and are presented in Tables 9.1 and 9.2.

In the case of the first set of frequencies, the values of λ obtained for the first four modes of vibration, for various values of γ and Δ , for a division of the beam into N = 2 and 3 segments are shown in Table 9.1 and are compared with the exact results obtained using the analysis presented in Chapter VIII. For, the second set, the values of λ obtained for the first four modes of vibration for N = 2 and 3 are shown in Table 9.2 and are compared with exact results. The exact results for the first and second sets were obtained using Eq.(8.45).

From Tables 9.1 and 9.2, it can be observed that, for all cases, the results obtained by finite element method even for very coarse subdivisions of the beam, are in excellent agreement with the exact ones. As stiffness and mass matrices including shear deformation and longitudinal inertia in addition to axial load and elastic foundation, involve double the number of degrees of freedom than those that exist if the secondary effects are neglected, twice as many natural frequencies result. In tables 9.1 and 9.2 the lower and higher spectrum of frequencies of simply supported beam are respectively listed. The second set of frequencies can also be observed to be in excellent agreement with the

TABLE - 9.1

Comparison of first set of values of λ for various values of Δ and γ from the Finite Element Method and those from exact analysis given in Chapter-VIII for a Simply Supported beam (K = 1.541, s = 0.046, d = 0.023).

Value of γ	Value of Δ	Mode No.	No. of Elements		Exact Results
			2	3	
0.0	3.0	I	12.3586	5.1254	4.7989
		II	33.9722	29.9049	29.7652
		III	101.0481	89.0871	65.9710
		IV	153.1285	142.7591	113.5342
2.0	3.0	I	11.3084	3.9253	3.2886
		II	34.3318	30.3129	28.3118
		III	101.1685	89.2232	65.4434
		IV	153.2073	142.8436	111.9216
2.0	0.0	I	23.2132	11.1546	10.2442
		II	42.5088	39.2334	35.7593
		III	108.1488	97.0813	73.8721
		IV	161.4194	151.3481	139.3192
4.0	3.0	I	8.4977	4.8672	4.3795
		II	35.1243	31.2071	25.9475
		III	101.4378	89.5272	63.8066
		IV	153.3832	143.0309	110.1274

TABLE - 9.2

Comparison of Second set of values of λ for various values of Δ and γ from the Finite Element Method and those from exact analysis given in Chapter - VIII for a Simply Supported beam ($K = 1.541$, $s = 0.046$, $d = 0.023$).

Value of γ	Value of Δ	Mode No.	No. of Elements		Exact Results.
			2	3	
0.0	2.0	I	962.7403	960.9861	842.969
		II	1006.2539	999.3401	874.078
		III	1093.2914	1071.8298	922.431
		IV	1191.2887	1164.5545	984.441
2.0	3.0	I	962.7403	960.9873	842.969
		II	1006.2539	999.3391	874.078
		III	1093.9256	1071.8298	922.427
		IV	1191.2887	1164.5545	984.433
2.0	0.0	I	962.7414	960.9839	842.970
		II	1006.2596	999.3436	874.081
		III	1093.3223	1071.8504	922.442
		IV	1191.3344	1164.6002	984.467
4.0	3.0	I	962.7403	960.9861	842.969
		II	1006.2539	999.3402	874.079
		III	1093.2937	1071.8309	922.432
		IV	1191.2887	1164.5545	984.442

exact ones. In Chapters IV and VIII these second set of frequencies are discussed in detail.

As ~~is~~ was mentioned previously, results for other boundary conditions can be easily obtained using the above stiffness and mass matrices with suitable changes in the Computer program and the data. The advantage of using the finite element method is that a beam with non-uniform section can also be analyzed by deviding the beam into a number of segments and assuming each segment has a constant cross section. This method provides us with an upper bound to the exact frequencies of the system and is quite general, satisfactorily encompassing all boundary conditions.

CHAPTER - X

NON-LINEAR TORSIONAL STABILITY OF LENGTHY THIN-WALLED BEAMS OF OPEN SECTION RESTING ON CONTINUOUS ELASTIC FOUNDATION*.

10.1. INTRODUCTION:

It is not uncommon, in structural design, to regard the elastic buckling load of a slender structural member as its failure load, and to pay little attention to its post-buckling behaviour. However, some structural members, such as simply supported thin plates loaded in compression, can support loads significantly greater than their elastic critical loads without deflecting excessively. This reserve of strength after buckling is due mainly to a redistribution of stress from the more flexible central area of the plate to the unloaded-edge regions (13). On the other hand, the load carrying capacity of some thin shell structures reduces rapidly after buckling. Such a structure is extremely sensitive to imperfections and disturbances, and may deform excessively at loads much less than its elastic critical load (45). Clearly, the post buckling behaviour of a structural member may have a decisive influence on the relation between its buckling and ultimate strengths.

The classical linear buckling theories (99) for elastic beams and columns necessarily predict buckling at loads that remain constant as the buckling amplitudes increase. Euler (99) first investigated the elastic flexural post-buckling behaviour of columns in 1744, by using the exact expression for curvature instead of the familiar small deflection approximation. This resulted in a post-buckling curve that rises so slowly that there is no significant increase in the load-carrying capacity until the deformations become gross.

The non-linear behaviour of members in uniform torsion was first investigated by Young (102) who considered circular cross sections. A related problem, the torsional stiffness of narrow rectangular sections under uniform axial tension, was examined by Buckley (14) and Weber (102) investigated the non-linear behaviour of narrow rectangular strips in pure torsion. Later, Cullimore (21) studied the behaviour of thin-walled I and Z sections. Weber and Cullimore showed that the torsional stiffness increases with the twist, and that this is due to a system of stresses acting along the helical fibres of the twisted member. The stress system is self equilibrating so that the outer fibres are in tension and the fibres closer to the twist axis are in compression.

Although Cullimore correctly derived the result for narrow rectangular members his expression for the non-linear torque component for I and Z sections is in doubt, because he used a constant lever arm, to obtain the torque contributed by the flange, instead of a variable lever arm, which is the distance from the twist axis to any point on the flange. Furthermore, his assumption of very thin walls leads to some inaccuracies when applied to the I and Z sections in common use. A more accurate theory of non-linear non-uniform torsion of thin-walled beams of open section is presented by Tso and Ghobarah (105) using the principle of minimum potential energy. Their theory takes into account the effect of large torsional deformation and allows very general loading and boundary conditions.

It can be seen that there is a surprising paucity of work on the elastic torsional post-buckling behaviour of doubly symmetric beams, in comparison with the extensive work on other structures (45). In particular, the behaviour of simply-supported and clamped beams and of I-section members resting on continuous elastic foundation has not been investigated. The purpose of the present Chapter, then, is to study theoretically the elastical torsional post-buckling behaviour of statically determinate beams of I-section resting on continuous Winkler type elastic foundation.

10.2. DEVELOPMENT OF GOVERNING DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS:

Consider a thin-walled beam of doubly-symmetric open cross section subject to axial compressive load. The relationship between the total torque T_t and the corresponding angle of twist ϕ in pure elastic torsion of a uniform thin-walled beam is given by Saint-Venant as:

$$T_t = GC_s \frac{d\phi}{dz} \qquad (10.1)$$

In the case of non-uniform torsion, Eq.(10.1) is extended to allow for the warping of the cross-sections of the beam; and

$$T_t = GC_s \frac{d\phi}{dz} - EC_w \frac{d^3\phi}{dz^3} \qquad (10.2)$$

The above Eq.(10.2) gives reasonable results for angles of twist approximately no greater than 5^0.

Experimental results obtained by Goodier (38) from tests have shown good qualitative, but poor quantitative, agreement with the theoretical conclusions from Eq.(10.2). If one examines the work of Weber (102), Gregory (42), Terrington (97) and Tso and Ghobarah (105), it can be seen that Eq.(10.2) is not complete insofar as there is a further torque component term to be considered. This term is due to the 'shortening effect' arising from torsion, described by Weber (102) and allowed for by Gregory (42) and, Tso and Ghobarah (105). Allowing for this component of torque, Eq.(10.2), becomes

$$T_t = GC_s \frac{d\phi}{dz} - EC_w \frac{d^3\phi}{dz^3} + 2EF\left(\frac{d\phi}{dz}\right)^3 \qquad (10.3)$$

where F is a constant dependent on cross sectional properties and is defined by

$$F \equiv I_R \left(I_{pc}/A\right)^2 \qquad (10.4)$$

in which I_{pc} is half the polar moment of inertia about the shear center and I_R the fourth moment of inertia about the shear center.

In the case of a thin-walled doubly symmetric I-beam of flange and web thicknesses t_f and t_w respectively; height between the centerlines of the flanges h, flange width b_f, and flange and web thicknesses being assumed as small compared with height h, i.e.

$t_f \ll h$, and $t_w \ll h$, the geometric properties in Eq.(10.4) can be evaluated as follows (105):

$$I_R = \frac{h^5 t_w}{320} + \frac{b_f h^4 t_f}{32} + \frac{b_f^5 t_f}{160} + \frac{b_f^3 h^2 t_f}{48} \tag{10.5}$$

and

$$I_{pc} = (1/24)\,(h^3 t_w + 2b_f^3 t_f + 6b_f h^2 t_f) \tag{10.6}$$

For a beam resting on a continuous Winkler type elastic foundation and subjected to an axial compressive load P, we have

$$\frac{dT_t}{dz} = \frac{PI_p}{A}\frac{d^2\phi}{dz^2} + K_t\,\phi \tag{10.7}$$

Substituting Eq.(10.3) in Eq.(10.7) the governing non-linear differential equation can be obtained as

$$EC_w\frac{d^4\phi}{dz^4} - 6EF\left(\frac{d\phi}{dz}\right)^2\frac{d^2\phi}{dz^2} - \left(GC_s - \frac{PI_p}{A}\right)\frac{d^2\phi}{dz^2} + K_t\,\phi = 0 \tag{10.8}$$

The boundary conditions associate with this problem are as follows:

(a) Simply supported end:

$$\phi = 0 \quad \text{and} \quad \frac{d^2\phi}{dz^2} = 0 \tag{10.9}$$

(b) Clamped end:

$$\phi = 0 \quad \text{and} \quad \frac{d\phi}{dz} = 0 \tag{10.10}$$

(c) Free end:

$$\frac{d^2\phi}{dz^2} = 0$$

and

$$EC_w \frac{d^3\phi}{dz^3} - 2EF\left(\frac{d\phi}{dz}\right)^3 - \left(GC_s - \frac{PI_p}{A}\right)\frac{d\phi}{dz} = 0 \qquad (10.11)$$

The general solution of Eq.(10.8) can be obtained by numerical methods using computer techniques. However, for the purpose of this thesis, approximate solutions are obtained for simply supported and clamped beams using Galerkin's method.

10.3. SIMPLY SUPPORTED BEAM:

For a beam simply supported at both ends, the boundary conditions are:

$$\phi = 0 \text{ and } \phi'' = 0 \text{ at } Z = 0 \qquad (10.12)$$

and

$$\phi = 0 \text{ and } \phi'' = 0 \text{ at } Z = 1 \qquad (10.13)$$

where primes denote differentiation with respect to the dimensionless length $Z = z/L$.

Eq.(10.8) can be written in non-dimensional form as:

$$\phi^{iv} - 6\overset{*}{\delta}(\phi')^2\phi'' - (K^2 - \Delta^2)\phi'' + 4\gamma^2\phi = 0 \qquad (10.14)$$

where

$$\overset{*}{\delta} = F/C_w \qquad (10.15)$$

To solve Eq.(10.14) by Galerkin's method, the angle of twist $\phi(Z)$ is assumed to be of the form

$$\phi(Z) = \overset{*}{\beta}\chi(Z) \qquad (10.16)$$

where $\overset{*}{\beta}$ is the torsional amplitude and χ is a function of Z. Since χ will be an approximate function assumed to satisfy the boundary conditions, by substituting Eq.(10.16) in Eq.(10.14), an error ϵ^{*} will be obtained as:

$$\epsilon^{*} = \beta^{*}\left[\chi^{iv} - 6\,\beta^{*2}\overset{*}{\delta}(\chi')^{2}\chi'' - (K^{2}-\Delta^{2})\chi'' + 4\gamma^{2}\chi\right] \qquad (10.17)$$

For minimizing the error ϵ^{*}, the Galerkin's Integral (79) is

$$\int_{0}^{1} \epsilon^{*}\chi \, dZ = 0 \qquad (10.18)$$

To satisfy the boundary conditions, Eqs.(10.12) and (10.13), we assume

$$\chi(Z) = \sin \pi Z \qquad (10.19)$$

Substituting Eqs.(10.17) and (10.19) into Eq.(10.18), we obtain the expression for the torsional post-buckling load for a simply supported beam as:

$$\Delta^{*2}_{cr} = K^{2} + \pi^{2} + 4\gamma^{2}/\pi^{2} + (3/2)\,\pi^{2}\,\overset{*}{\delta}\,\beta^{2} \qquad (10.20)$$

The corresponding linear torsional buckling load is given by (See (Eq.2.88)

$$\Delta^{2}_{cr} = K^{2} + \pi^{2} + 4\gamma^{2}/\pi^{2} \qquad (10.21)$$

Hence, the ratio of the non-linear buckling load to linear buckling load is given by

$$\frac{P^{*}}{P_{cr}} = \frac{\Delta^{*2}_{cr}}{\Delta^{2}_{cr}} = 1 + \frac{(3/2)\pi^{4}\,\overset{*}{\delta}\,\beta^{2}}{\left[\pi^{2}(K^{2}+\pi^{2}) + 4\gamma^{2}\right]} \qquad (10.22)$$

In the absence of elastic foundation, i.e., $\gamma = 0$, Eq.(10.22) reduces to

$$\frac{P^{*}}{P_{cr}} = \frac{\Delta^{*2}_{cr}}{\Delta^{2}_{cr}} = \left[1 + \frac{3\pi^2 \overset{*}{\delta} \beta^{*2}}{2(K^2+\pi^2)} \right] \qquad (10.23)$$

10.4. CLAMPED BEAM:

The boundary conditions for a beam clamped at both the ends are:

$$\phi = 0 \quad \text{and} \quad \phi' = 0 \quad \text{at } Z = 0 \qquad (10.24)$$

and

$$\phi = 0 \quad \text{and} \quad \phi' = 0 \quad \text{at } Z = 1 \qquad (10.25)$$

To satisfy the above conditions, the function $\chi(Z)$ can be assumed as:

$$\chi(Z) = \beta^{*}(1-\cos 2\pi Z) \qquad (10.26)$$

Substituting Eqs.(10.17) and (10.26) into Eq.(10.18) we obtain the expression for the torsional post-buckling load for a clamped beam as:

$$\Delta^{*2}_{cr} = K^2 + 4\pi^2 + 3\gamma^2/\pi^2 + 6\,\pi^2 \overset{*}{\delta} \beta^{*2} \qquad (10.27)$$

The corresponding linear torsional buckling load for a clamped beam is (See Eq.2.74)

$$\Delta^{2}_{cr} = K^2 + 4\pi^2 + 3\gamma^2/\pi^2 \qquad (10.28)$$

Hence, the ratio of the non-linear buckling load to linear buckling load is given by

$$\frac{P^*}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^{2}} = \left\{ 1 + \frac{6\pi^4 \delta^* \beta^{*2}}{\left[\pi^2(K^2+4\pi^2) + 3\nu^2\right]} \right\} \qquad (10.29)$$

In the absence of elastic foundation, ie., $\nu = 0$, Eq.(10.29) reduces to

$$\frac{P^*}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^{2}} = \left[1 + \frac{6\pi^2 \delta^* \beta^{*2}}{K^2 + 4\pi^2} \right] \qquad (10.30)$$